

# Wave operators, similarity and dynamics for a class of Schrödinger operators with generic non-mixed interface conditions in 1D.

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## Abstract

We consider a simple modification of the 1D-Laplacian where non-mixed interface conditions occurs at the boundaries of a finite interval. It has recently been shown that Schrödinger operators having this form allow a new approach to the transverse quantum transport through resonant heterostructures. In this perspective, it is important to control the deformations effects introduced on the spectrum and on the time propagator by this class of perturbations. In particular we are interested in uniform-in-time estimates of the perturbed semigroup.

The main difficulty is due to the non-selfadjoint character of our class of operators involving a lack of accretivity for the corresponding generator of the quantum dynamics. Our strategy consists in constructing stationary waves operators allowing to intertwine the modified non-selfadjoint Schrödinger operator with a corresponding 'physical' Hamiltonian. For small values of a deformation parameter ' $\theta$ ', this yields a dynamical comparison between the two models showing that the distance between the corresponding semigroups is dominated by  $|\theta|$  uniformly in time in the  $L^2$ -operator norm.

When a suitable energy constraint condition is assumed for the initial states, the above analysis adapts to the modelling of 1D quantum systems in the regime of quantum wells in a semiclassical island. In this framework, we show that artificial interface conditions introduce perturbations on the dynamics which are controlled by the ratio  $|\theta|/h^2$ , being  $h > 0$  a parameter fixing the 'quantum scale' of the system.

## 1 Introduction.

Schrödinger operators with non-mixed interface conditions have been recently considered in [9], by introducing the modified 1D Laplacian  $\Delta_\theta$

$$\left\{ \begin{array}{l} D(\Delta_\theta) = \left\{ u \in H^2(\mathbb{R} \setminus \{a, b\}) : \left[ \begin{array}{l} e^{-\frac{\theta}{2}} u(b^+) = u(b^-); \quad e^{-\frac{3}{2}\theta} u'(b^+) = u'(b^-) \\ e^{-\frac{\theta}{2}} u(a^-) = u(a^+); \quad e^{-\frac{3}{2}\theta} u'(a^-) = u'(a^+) \end{array} \right] \right\}, \\ \Delta_\theta u(x) = u''(x) \quad \text{for } x \in \mathbb{R} \setminus \{a, b\}. \end{array} \right. \quad (1.1)$$

where  $u(x^\pm)$  respectively denote the right and left limit of the function  $u$  in  $x$ . For all  $\theta \in \mathbb{C} \setminus \{0\}$ , the boundary conditions in (1.1) are non-selfadjoint and  $\Delta_\theta$  describes a singularly perturbed Laplacian, with non-selfadjoint point interactions acting in the boundary points  $\{a, b\}$ .

The interest in quantum models arising from  $\Delta_\theta$  stands upon the fact that a sharp exterior complex dilation, depending on  $\theta = i\tau$  with  $\tau > 0$ , maps  $-i\Delta_\theta$  into the accretive operator:  $-ie^{-2\theta} 1_{\mathbb{R} \setminus (a,b)}(x) \Delta_{2\theta}$ , where  $1_D$  denotes the characteristic function of the domain  $D$  (e.g. in Lemma 3.1 in [9]). For a short-range potential  $\mathcal{V}$  (i.e.:  $\mathcal{V} \in L^1$ ) compactly supported in  $(a, b)$ , the corresponding complex deformed Schrödinger operator

$$\mathcal{H}_\theta(\mathcal{V}, \theta) = -e^{-2\theta} 1_{\mathbb{R} \setminus (a,b)}(x) \Delta_{2\theta} + \mathcal{V}, \quad \text{supp } \mathcal{V} = (a, b), \quad (1.2)$$

is the generator of a contraction semigroup, while, in the case of time dependent potentials, uniform-in-time estimates hold for the corresponding dynamical system. According to the complex dilation technique, introduced in [1], [2], the quantum resonances of the undeformed operator  $\mathcal{H}_\theta(\mathcal{V})$

$$\mathcal{H}_\theta(\mathcal{V}) = -\Delta_\theta + \mathcal{V}, \quad (1.3)$$

are detected by exterior complex dilations and identify with the spectral points of  $\mathcal{H}_\theta(\mathcal{V}, \theta)$  in a suitable sector of the second Riemann sheet. Then, the adiabatic evolution problem for the resonant states of  $\mathcal{H}_\theta(\mathcal{V})$  rephrases, through an exterior complex dilation, into the adiabatic evolution problem for the corresponding eigenstates of  $\mathcal{H}_\theta(\mathcal{V}, \theta)$ . In this framework, accounting the contractivity property of the semigroup  $e^{-it\mathcal{H}_\theta(\mathcal{V}, \theta)}$  a 'standard'

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adiabatic theory can be developed (e.g. in [20]). This approach has been introduced in [9] where an adiabatic theorem is obtained for shape resonances in the regime of quantum wells in a semiclassical island.

Shape resonances play a central rôle in the mathematical analysis of semiconductor heterostructures (like tunneling diodes or possibly more complex structures; see [4], [5], [6], and references therein). In particular, the adiabatic approximation of the quantum evolution appears to be a key point in the derivation of reduced models for the dynamics of transverse quantum transport with concentrated non-linearities (e.g. in [13], [23], [24]; see also [10] for a linearized case). The purpose of this work is to justify, under suitable conditions, the use of Hamiltonians of the type  $\mathcal{H}_\theta(\mathcal{V})$  in the modelling of quantum transport through resonant heterostructures.

The relevance of the artificial interface conditions (1.1) in the description of physical quantum systems, stands upon the fact that they are expected to introduce small errors, w.r.t. the selfadjoint case, controlled by  $|\theta|$ . The quantum dynamics generated by  $\Delta_\theta$  has been considered in ([9] Proposition 2.2), where an asymptotic analysis of the time propagator  $e^{-it\Delta_\theta}$  (allowed by the explicit character of the operators) yields the expansion

$$e^{-it\Delta_\theta} = e^{-it\Delta} + \mathcal{R}(t, \theta), \quad (1.4)$$

holding in a suitable neighbourhood:  $|\theta| < \delta$ . Here, the reminder  $\mathcal{R}(t, \theta)$  is strongly continuous w.r.t.  $t$  and  $\theta$ , exhibits the group property w.r.t. the time variable and is such that

$$\sup_{t \in \mathbb{R}} \|\mathcal{R}(t, \theta)\|_{\mathcal{L}(L^2(\mathbb{R}))} = \mathcal{O}(|\theta|). \quad (1.5)$$

Thus, for  $\theta$  small enough,  $\Delta_\theta$  generates a group, strongly continuous both w.r.t.  $t$  and  $\theta$ , and allowing uniform-in-time estimates. In the perspective of modelling realistic physical situations through the modified Schrödinger operators  $\mathcal{H}_\theta(\mathcal{V})$ , an important step would consist in extending to this class of operators the expansion obtained in (1.4) for  $\mathcal{H}_\theta(0)$ . A possible approach considers  $\mathcal{H}_\theta(\mathcal{V})$  as a (selfadjoint) perturbation of the modified Laplacian:  $\mathcal{H}_\theta(0) = -\Delta_\theta$ ; it is worthwhile to notice that this would give a weaker result. For instance, implementing a Picard iteration on the Duhamel formula

$$u_t(\theta) = e^{-it\Delta_\theta} u_0 + i \int_0^t e^{-i(t-s)\Delta_\theta} \mathcal{V} u_s(\theta) ds, \quad (1.6)$$

and making use of the expansion (1.4), yields, in the case of a bounded potential  $\mathcal{V} \in L^\infty$ , the time dependent estimates

$$\|u_t(\theta)\|_{L^2(\mathbb{R})} \leq C_1 \|u_0\|_{L^2(\mathbb{R})} e^{C_2 \|\mathcal{V}\|_{L^\infty(\mathbb{R})} t}, \quad (1.7)$$

$$\|u_t(\theta) - u_t(0)\|_{L^2(\mathbb{R})} \leq C_3 |\theta| \|u_0\|_{L^2(\mathbb{R})} t e^{C_4 \|\mathcal{V}\|_{L^\infty(\mathbb{R})} t} \quad (1.8)$$

where  $C_i$ ,  $i = 1, \dots, 4$ , are suitable positive constants. It follows that, for an initial state  $u_0$ , the corresponding mild solution to the quantum evolution problem,  $u_t(\theta)$ , is, at each  $t$ ,  $L^2$ -norm Lipschitz continuous w.r.t.  $\theta$ , with a Lipschitz constant bounded by an exponentially increasing function of time. As an aside, we notice that the estimate (1.7) may also be obtained as a consequence of the Hille-Yoshida-Phillips Theorem, by using the second resolvent formula for  $(\mathcal{H}_\theta(\mathcal{V}) - z)^{-1}$  and resolvent estimates for  $(\mathcal{H}_\theta(0) - z)^{-1}$  arising from (1.4).

The relation (1.8) yields a finite-time control, depending on  $\|\mathcal{V}\|_{L^\infty(\mathbb{R})}$ , of the error introduced on the quantum evolution by the interface conditions. However, when  $\mathcal{V}$  describes the (possibly non-linear) interactions involving charge carriers in resonant heterostructures, its norm  $\|\mathcal{V}\|_{L^\infty(\mathbb{R})}$  is expected to be small compared to the energy of the particles, while the quantum evolution of relevant observables is characterized by a long time scale, corresponding to the inverse of the imaginary part of the shape resonances (examples of this mechanism are exhibited in [23] and [10]). In this framework, the use of modified Hamiltonians of the type  $\mathcal{H}_\theta(\mathcal{V})$  would be justified by a stronger uniform-in-time estimate for the error  $\|u_t(\theta) - u_t(0)\|_{L^2(\mathbb{R})}$  as  $\theta \rightarrow 0$ .

Adopting a different approach, in what follows the operator  $\mathcal{H}_\theta(\mathcal{V})$  is considered as a non-selfadjoint perturbation of the selfadjoint Hamiltonian  $\mathcal{H}_0(\mathcal{V})$ . Non-selfadjoint perturbations of the type  $T(x) = T + xAB^*$  have been studied in [14] where, using suitable smoothness assumptions on  $A$  and  $B$ , the 'stationary' waves operators for the couple  $\{T(x), T\}$  are given and the corresponding similarity between  $T$  and  $T(x)$  is exploited to define the dynamics generated by  $-iT(x)$ . Here we adapt the same strategy to the case where  $T = \mathcal{H}_0(\mathcal{V})$ , while the perturbation is determined by generic, non-mixed, interface conditions occurring at the boundaries of the potential's support. This is a larger class of operators, parametrized by a couple of complex, which includes both the cases of  $\mathcal{H}_\theta(\mathcal{V})$  and  $(\mathcal{H}_\theta(\mathcal{V}))^*$ . In this extended framework, an accurate resolvent analysis and Krein's like formula for generalized eigenfunctions, allows to obtain a small- $\theta$  expansion of the 'stationary waves operators'. Then the quantum evolution group generated by  $-i\mathcal{H}_\theta(\mathcal{V})$  is determined by conjugation from  $e^{-it\mathcal{H}_0(\mathcal{V})}$  and an uniform-in-time estimate for the 'distance' between the two dynamics is obtained (see Theorem 1.2).

Similarity transformations, from non-selfadjoint to similar selfadjoint operators, have been recently studied in [15], where the authors focus on the particular case of Schrödinger operators defined with non-selfadjoint

Robin-type conditions occurring at the boundary of an interval. In the case of parity and time-reversal symmetry ( $\mathcal{PT}$ -symmetry), the similarity of this model with a selfadjoint Hamiltonian is derived. It is worth noticing that, when  $\theta \in i\mathbb{R}$ , the modified Laplacian  $\Delta_\theta$  actually exhibits the  $\mathcal{PT}$ -symmetry (once the parity is defined with respect to the point  $(a+b)/2$ ). However, the models introduced in the next sections are generically not  $\mathcal{PT}$ -symmetric (see the definitions (1.11) and (1.18) below).

In the second part of this work, we focus on Schrödinger operators modelling one quantum particle in the regime of quantum wells in a semiclassical island. Our aim is to obtain a comparison between the dynamical system modified by non-mixed interface conditions and the unitary dynamics, generated by the corresponding selfadjoint Hamiltonian. In particular, we are interested in the limit when the 'quantum scale' parameter  $h$  of the model goes to zero. A direct extension of the previous analysis, in this framework, seems to be impossible. Actually, a simple rescaling of the problem w.r.t.  $h$  would lead to Weyl's function estimates which are exponentially growing w.r.t.  $h$  as  $h \rightarrow 0$  (see Section 4). Nevertheless we show that, if the initial state is imbedded in a suitable spectral subspace, then the  $L^2$ -distance between the modified and the physical dynamics is dominated, for each  $h > 0$ , by the sum  $\sum_{i=1,2} |\theta_i|/h^2$  uniformly in time (see Theorem 1.3).

The above mentioned models are introduced in the next Subsections where the main results are stated.

## 1.1 Notation

We use a generalization of the Landau notation  $\mathcal{O}(\cdot)$ :

**Definition 1.1** *Let be  $X$  a metric space and  $f, g : X \rightarrow \mathbb{C}$ . Then  $f = \mathcal{O}(g) \stackrel{def}{\iff} \forall x \in X$  it holds:  $f(x) = p(x)g(x)$ , being  $p$  a bounded map  $X \rightarrow \mathbb{C}$ .*

The following notation are also adopted

$\mathcal{B}_\delta(p)$  is the open ball of radius  $\delta$  centered in a point  $p \in \mathbb{C}$ .

$1_\Omega(\cdot)$  is the characteristic function of a domain  $\Omega$ .

$C_x^k(U)$  is the set of  $C^k$ -continuous functions w.r.t.  $x \in U \subseteq \mathbb{R}$ .

$\mathcal{H}_z(D)$  is the set of holomorphic functions w.r.t.  $z \in D \subseteq \mathbb{C}$ .

$\partial_j f(x_1, \dots, x_n)$ ,  $j \in \{1, \dots, n\}$ , denotes the derivative of  $f$  w.r.t. the variable  $x_j$ .

The notation ' $\lesssim$ ' or ' $\gtrsim$ ', appearing in some of the following proofs, denote the inequalities: ' $\leq C$ ' or ' $\geq C$ ' being  $C$  a suitable positive constant.

## 1.2 Schrödinger operators with non-mixed interface conditions.

We consider the family of modified Schrödinger operators  $Q_{\theta_1, \theta_2}(\mathcal{V})$ , depending on a couple of complex parameters,  $(\theta_1, \theta_2) \in \mathbb{C}^2$ , and on a selfadjoint short-range potential, compactly supported over the interval  $(a, b) \subset \mathbb{R}$ ,

$$\mathcal{V} \in L^1(\mathbb{R}, \mathbb{R}), \quad \text{supp } \mathcal{V} = (a, b). \quad (1.9)$$

The parameters  $\theta_1$  and  $\theta_2$  fix the interface conditions,

$$\begin{cases} e^{-\frac{\theta_1}{2}} u(b^+) = u(b^-), & e^{-\frac{\theta_2}{2}} u'(b^+) = u'(b^-), \\ e^{-\frac{\theta_1}{2}} u(a^-) = u(a^+), & e^{-\frac{\theta_2}{2}} u'(a^-) = u'(a^+), \end{cases} \quad (1.10)$$

occurring at the boundary of the potential's support and  $Q_{\theta_1, \theta_2}(\mathcal{V})$  is defined as follows

$$Q_{\theta_1, \theta_2}(\mathcal{V}) : \begin{cases} D(Q_{\theta_1, \theta_2}(\mathcal{V})) = \{u \in H^2(\mathbb{R} \setminus \{a, b\}) \mid (1.10) \text{ holds} \}, \\ (Q_{\theta_1, \theta_2}(\mathcal{V})u)(x) = -u''(x) + \mathcal{V}(x)u(x), \quad x \in \mathbb{R} \setminus \{a, b\}. \end{cases} \quad (1.11)$$

The set  $\{Q_{\theta_1, \theta_2}(\mathcal{V}), (\theta_1, \theta_2) \in \mathbb{C}^2\}$  is closed w.r.t. the adjoint operation: a direct computation shows that

$$(Q_{\theta_1, \theta_2}(\mathcal{V}))^* = Q_{-\theta_2^*, -\theta_1^*}(\mathcal{V}). \quad (1.12)$$

The subset of selfadjoint operators in this class is identified by the conditions: for  $\theta_j = r_j e^{i\varphi_j}$ ,  $j = 1, 2$ ,

$$\begin{cases} \varphi_1 + \varphi_2 = \pi + 2\pi k, & k \in \mathbb{Z}, \\ r_1 = r_2. \end{cases} \quad (1.13)$$

When (1.13) are not satisfied, the corresponding operator  $Q_{\theta_1, \theta_2}(\mathcal{V})$  is neither selfadjoint nor symmetric, since in this case:  $Q_{\theta_1, \theta_2}(\mathcal{V}) \notin (Q_{\theta_1, \theta_2}(\mathcal{V}))^*$ .

For each couple  $\{\theta_1, \theta_2\}$ ,  $Q_{\theta_1, \theta_2}(\mathcal{V})$  identifies with a, possibly non-selfadjoint, extension of the Hermitian operator  $Q^0(\mathcal{V})$

$$D(Q^0(\mathcal{V})) = \{u \in H^2(\mathbb{R}) \mid u(0) = u'(0) = 0\}, \quad (1.14)$$

and defines an explicitly solvable model w.r.t. the selfadjoint Hamiltonian  $Q_{0,0}(\mathcal{V})$ . Non-selfadjoint models arising from proper extensions of Hermitian operators with gaps have been already considered in literature, for instance in [8], [28] (see also [17]–[19] and [7] for the general case of adjoint pairs of operators). In these works, the formalism of *boundary triples* (e.g. in [16] for adjoint pairs) is adopted; this leads to Krein-like formulas expressing the resolvent of an extended operator in terms of the resolvent of a 'reference' extension plus a finite rank part depending on the Weyl function of the triple. In Section 2 we give Krein-like formulas for the difference  $(Q_{\theta_1, \theta_2}(\mathcal{V}) - z)^{-1} - (Q_{0,0}(\mathcal{V}) - z)^{-1}$ . Exploiting this framework, we show that  $Q_{\theta_1, \theta_2}(\mathcal{V})$  is an analytic family in the sense of Kato both w.r.t. the variables  $\theta_1$  and  $\theta_2$  and study its spectral profile depending on  $\mathcal{V}$ . The result is exposed in the Proposition 2.4; in particular, for positive defined  $\mathcal{V}$ , we obtain:  $\sigma(Q_{\theta_1, \theta_2}(\mathcal{V})) = \sigma_{ac}(Q_{0,0}(\mathcal{V})) = \mathbb{R}_+$ , provided that  $\theta_1$  and  $\theta_2$  are small enough.

Under the same assumptions, in Section 3, a family of intertwining operators  $\mathcal{W}_{\theta_1, \theta_2}$  for the couple  $\{Q_{\theta_1, \theta_2}(\mathcal{V}), Q_{0,0}(\mathcal{V})\}$  is introduced as the analogous of the usual stationary waves operators in selfadjoint frameworks. Using the eigenfunctions expansion obtained in Subsection 2.3, we get a small- $\theta_i$  expansion of  $\mathcal{W}_{\theta_1, \theta_2}$  allowing to define the quantum evolution group  $e^{-iQ_{\theta_1, \theta_2}(\mathcal{V})t}$  from  $e^{-iQ_{0,0}(\mathcal{V})t}$  by conjugation. Then, we develop a quantitative comparison showing that  $e^{-iQ_{\theta_1, \theta_2}(\mathcal{V})t} - e^{-iQ_{0,0}(\mathcal{V})t}$  is controlled by  $|\theta_i|$ ,  $i = 1, 2$ , uniformly in time, in the  $L^2$ -operator norm. The result is presented in the following theorem and the proof is given in Subsection 3.1. It can be adapted to the particular case of  $\mathcal{H}_\theta(\mathcal{V})$ , by noticing that:  $\mathcal{H}_\theta(\mathcal{V}) = Q_{\theta, 3\theta}(\mathcal{V})$ .

**Theorem 1.2** *Let  $\mathcal{V}$  fulfills the conditions (1.9) and*

$$\langle u, \mathcal{V}u \rangle_{L^2((a,b))} > 0 \quad \forall u \in L^2((a,b)), \quad (1.15)$$

*and assume  $(\theta_1, \theta_2) \in \mathcal{B}_\delta((0,0))$  with  $\delta > 0$  small enough. The operator  $-iQ_{\theta_1, \theta_2}(\mathcal{V})$  generates a strongly continuous group of bounded operators on  $L^2(\mathbb{R})$ ,  $\{e^{-itQ_{\theta_1, \theta_2}(\mathcal{V})}\}_{t \in \mathbb{R}}$ . For a fixed  $t$ ,  $e^{-itQ_{\theta_1, \theta_2}(\mathcal{V})}$  defines an analytic family of type A w.r.t.  $(\theta_1, \theta_2)$  and the expansion*

$$e^{-itQ_{\theta_1, \theta_2}(\mathcal{V})} = e^{-itQ_{0,0}(\mathcal{V})} + \mathcal{R}(t, \theta_1, \theta_2), \quad (1.16)$$

*holds with an uniformly bounded in time reminder s.t.*

$$\sup_{t \in \mathbb{R}} \|\mathcal{R}(t, \theta_1, \theta_2)\|_{\mathcal{L}(L^2(\mathbb{R}))} = \mathcal{O}(\theta_1) + \mathcal{O}(\theta_2). \quad (1.17)$$

The first part of this work concludes with the Subsection 3.2 where the scattering system related to the pair  $\{Q_{\theta_1, \theta_2}(\mathcal{V}), Q_{0,0}(\mathcal{V})\}$  and the existence of non-stationary waves operators is considered. A partial answer to this question is obtained under the assumption of small perturbations (i.e.  $\theta_{i=1,2}$  small): under the assumptions of the Theorem 1.2, it is shown that  $\mathcal{W}_{\theta_1, \theta_2}$  coincide with 'physical' waves operators, according to the time dependent definition (Lemma 3.2). Finally we mention possible further perspectives of this analysis concerning the more general situation where  $(\theta_1, \theta_2)$  is not close to the origin.

### 1.3 Modelling resonant heterostructures.

Let introduce the modified operators  $Q_{\theta_1, \theta_2}^h(\mathcal{V})$ , depending on the small parameter  $h \in (0, h_0)$ ,  $h_0 > 0$ , and defined according to

$$Q_{\theta_1, \theta_2}^h(\mathcal{V}) : \begin{cases} D(Q_{\theta_1, \theta_2}^h(\mathcal{V})) = \{u \in H^2(\mathbb{R} \setminus \{a, b\}) \mid (1.10) \text{ holds} \}, \\ (Q_{\theta_1, \theta_2}^h(\mathcal{V})u)(x) = -h^2 u''(x) + \mathcal{V}(x)u(x), \quad x \in \mathbb{R} \setminus \{a, b\}. \end{cases} \quad (1.18)$$

with  $\mathcal{V}$  locally supported on  $(a, b)$ . In the applications perspectives, rather relevant is the case of a positive and bounded  $\mathcal{V}^h$  formed by a potential barrier superposed to a collection of  $h$ -size supported potential wells (usually referred to as *quantum wells*). In this setting, the operators  $Q_{\theta_1, \theta_2}^h(\mathcal{V}^h)$  can be associated to one-particle quantum systems in the regime of quantum wells in a semiclassical island. Hamiltonians of this type have been introduced in [9] with the purpose of realizing models of electronic transverse transport through resonant heterostructures. When the initial state describes incoming charge carriers in the conduction band, the quantum dynamics of such systems is expected be driven by the, possibly non-linear, adiabatic evolution of a finite number of resonant states related to the shape resonances. This picture, arising in the physical literature, is confirmed by the analysis presented in [13], [23] concerning the case of a 1D Schrödinger-Poisson selfadjoint

model with a double barrier, and in [10], where an application involving Hamiltonians of the type  $Q_{\theta_1, \theta_2}^h(\mathcal{V})$  is considered.

Let  $h_0 > 0$  and  $\{U_{h,i}\}_{i=1}^N$  be a finite collection of disjoint compact subsets of  $(a, b)$  such that

$$\sup_{\substack{x, y \in U_{h,i} \\ i \leq N}} |x - y| \leq c_0 h, \quad (1.19)$$

for all  $h \in (0, h_0)$  and define a collection of *quantum wells*  $\mathcal{W}^h$  supported on  $U_{h,i}$  as

$$\mathcal{W}^h \in L^\infty(\mathbb{R}, \mathbb{R}), \quad \text{supp } \mathcal{W}^h = \bigcup_{i=1}^N U_{h,i}. \quad (1.20)$$

We assume that  $\mathcal{V}^h = \mathcal{V} + \mathcal{W}^h$  where  $\mathcal{W}^h$  is defined according to (1.19)-(1.20), while  $\mathcal{V}$  is a potential barrier fulfilling the conditions

$$\mathcal{V} \in L^\infty(\mathbb{R}, \mathbb{R}), \quad \text{supp } \mathcal{V} = (a, b), \quad 1_{(a,b)} \mathcal{V} > 0. \quad (1.21)$$

and

$$\inf_{(a,b)} (\mathcal{V} + \mathcal{W}^h) \geq 0. \quad (1.22)$$

Our aim is to compare the dynamics generated by the operators  $Q_{\theta_1, \theta_2}^h(\mathcal{V} + \mathcal{W}^h)$  and  $Q_{0,0}^h(\mathcal{V} + \mathcal{W}^h)$  when  $\theta_i$ ,  $i = 1, 2$ , are small. To this aim we introduce the subsets  $\Omega_c(\mathcal{V}) \subset \sigma(Q_{0,0}^h(\mathcal{V}))$  defined by the constrain

$$\Omega_c(\mathcal{V}) = \{k^2 \in \mathbb{R}_+ \mid \inf \mathcal{V} - k^2 > c\}, \quad (1.23)$$

and denote with  $\Pi_{\Omega_c(\mathcal{V})}$  the spectral projector on  $\Omega_c(\mathcal{V})$  associated with the operator  $Q_{0,0}^h(\mathcal{V})$ . In Section 4 we prove the following result.

**Theorem 1.3** *Let  $\mathcal{V}^h = \mathcal{V} + \mathcal{W}^h$  defined according to (1.19)-(1.20) and (1.21)-(1.22) and assume that  $\Omega_c(\mathcal{V}) \neq \emptyset$  for  $c > 0$ . Then, there exist  $h_0, \delta > 0$  such that, for any  $(\theta_1, \theta_2) \in B_\delta((0, 0))$  and  $h \in (0, h_0)$ , the operator  $-iQ_{\theta_1, \theta_2}^h(\mathcal{V}^h)$  generates a strongly continuous group of bounded operators  $e^{-itQ_{\theta_1, \theta_2}^h(\mathcal{V}^h)}$  from  $\Pi_{\Omega_c(\mathcal{V})}L^2(\mathbb{R})$  to  $L^2(\mathbb{R})$ . For a fixed  $t$ ,  $e^{-itQ_{\theta_1, \theta_2}^h(\mathcal{V}^h)}\Pi_{\Omega_c(\mathcal{V})}$  defines an analytic family of type A w.r.t.  $(\theta_1, \theta_2)$  and the expansion*

$$\left(e^{-itQ_{\theta_1, \theta_2}^h(\mathcal{V}^h)} - e^{-itQ_{0,0}^h(\mathcal{V}^h)}\right)\Pi_{\Omega_c(\mathcal{V})} = \mathcal{R}^h(t, \theta_1, \theta_2), \quad (1.24)$$

holds with

$$\sup_{t \in \mathbb{R}} \|\mathcal{R}^h(t, \theta_1, \theta_2)\|_{\mathcal{L}(L^2(\mathbb{R}))} = \sum_{i=1,2} \mathcal{O}\left(\frac{\theta_i}{h^2}\right).$$

## 2 Boundary triples and Krein-like resolvent formulas.

Point perturbation models, as  $Q_{\theta_1, \theta_2}(\mathcal{V})$ , can be described as restrictions of a *larger* operator through linear boundary relations. Let introduce  $Q(\mathcal{V})$

$$\begin{cases} D(Q(\mathcal{V})) = H^2(\mathbb{R} \setminus \{a, b\}), \\ (Q(\mathcal{V})u)(x) = -u''(x) + \mathcal{V}(x)u(x) \quad \text{for } x \in \mathbb{R} \setminus \{a, b\}, \end{cases} \quad (2.1)$$

with  $\mathcal{V}$  defined according to (1.9), and let  $Q^0(\mathcal{V})$  be such that:  $(Q^0(\mathcal{V}))^* = Q(\mathcal{V})$ . Explicitly,  $Q^0(\mathcal{V})$  identifies with the symmetric restriction of  $Q(\mathcal{V})$  to the domain  $D(Q^0(\mathcal{V})) = \{u \in D(Q(\mathcal{V})) \mid u(\alpha) = u'(\alpha) = 0 \ \forall \alpha \in \{a, b\}\}$ . The related defect spaces,  $\mathcal{N}_z = \ker(Q(\mathcal{V}) - z)$ , are 4-dimensional subspaces of  $D(Q(\mathcal{V}))$  generated, for  $z \in \mathbb{C} \setminus \mathbb{R}$ , by the independent solutions of the problem

$$\begin{cases} (-\partial_x^2 + \mathcal{V} - z)u(x) = 0, & x \in \{a, b\}, \\ u \in D(Q(\mathcal{V})). \end{cases} \quad (2.2)$$

A *boundary triple*  $\{\mathbb{C}^4, \Gamma_1, \Gamma_2\}$  for  $Q(\mathcal{V})$  is defined with two linear boundary maps  $\Gamma_{i=1,2} : D(Q(\mathcal{V})) \rightarrow \mathbb{C}^4$  fulfilling, for any  $\psi, \varphi \in D(Q(\mathcal{V}))$ , the equation

$$\langle \psi, Q(\mathcal{V})\varphi \rangle_{L^2(\mathbb{R})} - \langle Q(\mathcal{V})\psi, \varphi \rangle_{L^2(\mathbb{R})} = \langle \Gamma_1\psi, \Gamma_2\varphi \rangle_{\mathbb{C}^4} - \langle \Gamma_2\psi, \Gamma_1\varphi \rangle_{\mathbb{C}^4}, \quad (2.3)$$

and such that the transformation  $(\Gamma_1, \Gamma_2) : D(Q(\mathcal{V})) \rightarrow \mathbb{C}^4 \times \mathbb{C}^4$  is surjective. A proper extension  $Q_{ext}$  of  $Q^0(\mathcal{V})$  is called *almost solvable* if there exists a boundary triple  $\{\mathbb{C}^4, \Gamma_1, \Gamma_2\}$  and a matrix  $M \in \mathbb{C}^{4,4}$  such that it coincides with the restriction of  $Q(\mathcal{V})$  to the domain:  $\{u \in D(Q(\mathcal{V})) \mid M\Gamma_1 u = \Gamma_2 u\}$ . Using the notation  $Q_{ext} = Q_M(\mathcal{V})$ , the following characterization holds (e.g. [28], Theorem 1.1)

$$Q^0(\mathcal{V}) \subset Q_M(\mathcal{V}) \subset Q(\mathcal{V}), \quad (Q_M(\mathcal{V}))^* = Q_{M^*}(\mathcal{V}). \quad (2.4)$$

Let the  $\tilde{Q}(\mathcal{V})$  denotes the particular restriction of  $Q(\mathcal{V})$  associated to the conditions:  $\Gamma_1 u = 0$ , i.e.

$$D(\tilde{Q}(\mathcal{V})) = \{u \in D(Q(\mathcal{V})) \mid \Gamma_1 u = 0\}, \quad (2.5)$$

and introduce the linear maps  $\gamma(z, \mathcal{V})$  and  $q(z, \mathcal{V})$  as follows

$$\gamma(z, \mathcal{V}) = (\Gamma_1|_{\mathcal{N}_z})^{-1}, \quad q(z, \mathcal{V}) = \Gamma_2 \circ \gamma(z, \mathcal{V}), \quad (2.6)$$

where  $\Gamma_1|_{\mathcal{N}_z}$  is the restriction of  $\Gamma_1$  to  $\mathcal{N}_z$ . According to the relation (2.3),  $\tilde{Q}(\mathcal{V})$  is selfadjoint and  $\mathbb{C} \setminus \mathbb{R} \subset \rho(\tilde{Q}(\mathcal{V}))$ . For  $z \in \mathbb{C} \setminus \mathbb{R}$ ,  $\gamma(\cdot, z, \mathcal{V})$  and  $q(z, \mathcal{V})$  define holomorphic families of bounded operators in  $\mathcal{L}(\mathbb{C}^4, L^2(\mathbb{R}))$  and  $\mathcal{L}(\mathbb{C}^4, \mathbb{C}^4)$ , allowing an analytic continuation to the resolvent set  $\rho(\tilde{Q}(\mathcal{V}))$  (e.g. in [22]). The maps  $\gamma(\cdot, z, \mathcal{V})$  and  $q(z, \mathcal{V})$  are respectively referred to as the *Gamma field* and the *Weyl function* associated to the triple  $\{\mathbb{C}^4, \Gamma_1, \Gamma_2\}$ . With this formalism, a resolvent formula expresses the difference  $(Q_M(\mathcal{V}) - z)^{-1} - (\tilde{Q}(\mathcal{V}) - z)^{-1}$  in terms of finite rank operator with range  $\mathcal{N}_z$

$$(Q_M(\mathcal{V}) - z)^{-1} - (\tilde{Q}(\mathcal{V}) - z)^{-1} = \gamma(z, \mathcal{V}) (M - q(z, \mathcal{V}))^{-1} \gamma^*(\bar{z}, \mathcal{V}), \quad z \in \rho(Q_M(\mathcal{V})) \cap \rho(\tilde{Q}(\mathcal{V})) \quad (2.7)$$

(e.g. in [28], Theorem 1.2). In many situations, the interface conditions occurring in the points  $\{a, b\}$  can be also represented in the form:  $A\Gamma_1 u = B\Gamma_2 u$ , where  $A, B \in \mathbb{C}^{4,4}$ . We denote with  $Q_{A,B}(\mathcal{V})$  the corresponding restriction

$$\begin{cases} D(Q_{A,B}(\mathcal{V})) = \{u \in D(Q(\mathcal{V})) \mid A\Gamma_1 u = B\Gamma_2 u\}, \\ Q_{A,B}(\mathcal{V}) u = Q(\mathcal{V}) u. \end{cases} \quad (2.8)$$

With this parametrization, we have:  $\tilde{Q}(\mathcal{V}) = Q_{1,0}(\mathcal{V})$ , while the resolvent's formula rephrases as

$$(Q_M(\mathcal{V}) - z)^{-1} - (\tilde{Q}(\mathcal{V}) - z)^{-1} = -\gamma(z, \mathcal{V}) \left[ (Bq(z, \mathcal{V}) - A)^{-1} B \right] \gamma^*(\bar{z}, \mathcal{V}), \quad z \in \rho(Q_M(\mathcal{V})) \quad (2.9)$$

In the perspective of a comparison between the quantum models arising from  $Q_{\theta_1, \theta_2}(\mathcal{V})$  and  $Q_{0,0}(\mathcal{V})$  a natural choice is

$$\Gamma_1 u = \begin{pmatrix} u'(b^-) - u'(b^+) \\ u(b^+) - u(b^-) \\ u'(a^-) - u'(a^+) \\ u(a^+) - u(a^-) \end{pmatrix}, \quad \Gamma_2 u = \frac{1}{2} \begin{pmatrix} u(b^+) + u(b^-) \\ u'(b^+) + u'(b^-) \\ u(a^+) + u(a^-) \\ u'(a^+) + u'(a^-) \end{pmatrix}, \quad (2.10)$$

which leads to:  $\tilde{Q}(\mathcal{V}) = Q_{0,0}(\mathcal{V})$ . According to the definitions (1.11) and (2.1), the operator  $Q_{\theta_1, \theta_2}(\mathcal{V})$  identifies with the restriction of  $Q(\mathcal{V})$  parametrized by the  $\mathbb{C}^{4,4}$ -block-diagonal matrices

$$A_{\theta_1, \theta_2} = \begin{pmatrix} a(\theta_1, \theta_2) & & & \\ & & & \\ & & a(-\theta_1, -\theta_2) & \\ & & & \end{pmatrix}, \quad B_{\theta_1, \theta_2} = \begin{pmatrix} b(\theta_1, \theta_2) & & & \\ & & & \\ & & b(-\theta_1, -\theta_2) & \\ & & & \end{pmatrix}, \quad (2.11)$$

defined with

$$a(\theta_1, \theta_2) = \begin{pmatrix} 1 + e^{\frac{\theta_2}{2}} & 0 \\ 0 & 1 + e^{\frac{\theta_1}{2}} \end{pmatrix}, \quad b(\theta_1, \theta_2) = 2 \begin{pmatrix} 0 & 1 - e^{\frac{\theta_2}{2}} \\ e^{\frac{\theta_1}{2}} - 1 & 0 \end{pmatrix}. \quad (2.12)$$

Using (2.10) and (2.12)-(2.12), the nonselfadjoint boundary relations (1.10) rephrase as

$$A_{\theta_1, \theta_2} \Gamma_1 u = B_{\theta_1, \theta_2} \Gamma_2 u, \quad (2.13)$$

which leads to the equivalent definition

$$Q_{\theta_1, \theta_2}(\mathcal{V}) : \begin{cases} D(Q_{\theta_1, \theta_2}(\mathcal{V})) = \{u \in D(Q(\mathcal{V})) \mid A_{\theta_1, \theta_2} \Gamma_1 u = B_{\theta_1, \theta_2} \Gamma_2 u\}, \\ Q_{\theta_1, \theta_2}(\mathcal{V}) u = Q(\mathcal{V}) u. \end{cases} \quad (2.14)$$

In this framework, the relation (2.9) explicitly writes as

$$(Q_{\theta_1, \theta_2}(\mathcal{V}) - z)^{-1} = (Q_{0,0}(\mathcal{V}) - z)^{-1} - \sum_{i,j=1}^4 \left[ (B_{\theta_1, \theta_2} q(z, \mathcal{V}) - A_{\theta_1, \theta_2})^{-1} B_{\theta_1, \theta_2} \right]_{ij} \langle \gamma(e_j, \bar{z}, \mathcal{V}), \cdot \rangle_{L^2(\mathbb{R})} \gamma(e_i, z, \mathcal{V}), \quad (2.15)$$

where  $\{e_i\}_{i=1}^4$  is the standard basis in  $\mathbb{C}^4$ , while  $\gamma(v, z, \mathcal{V})$  denotes the action of  $\gamma(z, \mathcal{V})$  on the vector  $v$ . The corresponding integral kernel,  $\mathcal{G}_{\theta_1, \theta_2}^z(x, y)$ , is

$$\mathcal{G}_{\theta_1, \theta_2}^z(x, y) = \mathcal{G}_{0,0}^z(x, y) - \sum_{i,j=1}^4 \left[ (B_{\theta_1, \theta_2} q(z, \mathcal{V}) - A_{\theta_1, \theta_2})^{-1} B_{\theta_1, \theta_2} \right]_{ij} \gamma(e_j, y, z, \mathcal{V}) \gamma(e_i, x, z, \mathcal{V}), \quad (2.16)$$

## 2.1 The Jost's solutions.

In order to obtain explicit representations of the operators  $\gamma(\cdot, z, \mathcal{V})$  and  $q(z, \mathcal{V})$  appearing at the r.h.s. of (2.15), it is necessary to define a particular basis of the defect spaces  $\mathcal{N}_z$ . A possible choice is given in terms of the Green's function of the operator  $(Q_{0,0}(\mathcal{V}) - z)$  and of their derivatives. This motivates the forthcoming analysis, where the properties of the functions in  $\mathcal{N}_z$  are investigated by using the Jost's solutions associated to  $Q_{0,0}(\mathcal{V})$ . Our aim is to provide with explicit low and high energy asymptotic in the case of compactly supported and positive defined potentials. We follow a standard approach adapting arguments from one dimensional scattering to this particular case. Detailed computations, for selfadjoint one-dimensional Schrödinger operators with generic short range potentials, are presented in [29].

Consider the problem

$$(-\partial_x^2 + \mathcal{V})u = \zeta^2 u, \quad \text{for } x \in \mathbb{R} \text{ and } \zeta \in \mathbb{C}^+. \quad (2.17)$$

The Jost solutions of (2.17),  $\chi_{\pm}$ , are respectively defined by the exterior conditions

$$\chi_+|_{x>b} = e^{i\zeta x}, \quad \chi_-|_{x<a} = e^{-i\zeta x}. \quad (2.18)$$

In the next proposition we resume known properties of the functions  $\chi_{\pm}$  (see the Lemmata 4.1.1, 4.3.1 and 5.1.1 in [29] for the proof).

**Proposition 2.1** *Let  $\mathcal{V}$  be defined according to (1.9). The solutions  $\chi_{\pm}$  of the problem (2.17)-(2.18) belong to  $\mathcal{C}_x^1(\mathbb{R}, \mathcal{H}_{\zeta}(\mathbb{C}^+))$  having continuous extension to the real axis. For  $\zeta \in \mathbb{C}^+$ , the relations*

$$\chi_{\pm}(x, \zeta) = e^{\pm i\zeta x} \mathcal{O}(1), \quad \partial_x \chi_{\pm}(x, \zeta) = e^{\pm i\zeta x} \mathcal{O}(1 + \zeta), \quad (2.19)$$

hold with  $\mathcal{O}(\cdot)$  referred to the metric space  $\mathbb{R} \times \overline{\mathbb{C}^+}$ .

The Jost function corresponding to  $\chi_{\pm}(\cdot, \zeta)$ , denoted in the following with  $w(\zeta)$ , is defined as the Wronskian associated to the couple  $\chi_{\pm}(\cdot, \zeta)$ . Setting

$$w(f, g) = fg' - f'g, \quad (2.20)$$

we have

$$w(\zeta) = \chi_+(\cdot, \zeta) \partial_1 \chi_-(\cdot, \zeta) - \partial_1 \chi_+(\cdot, \zeta) \chi_-(\cdot, \zeta) \quad (2.21)$$

According to the definition of  $\chi_{\pm}$ , this function is independent from the space variable, while due to the result of Proposition 2.1,  $w(\zeta)$  is holomorphic w.r.t.  $\zeta$  in an open half complex plane including  $\overline{\mathbb{C}^+}$ . The point spectrum of  $Q_{0,0}(\mathcal{V})$  is defined by the solutions  $z = \zeta^2$  of the problem:  $w(\zeta) = 0$ ,  $\zeta \in \mathbb{C}^+$  (e.g. in [29], Chp.5). Since  $Q_{0,0}(\mathcal{V})$  is a selfadjoint Schrödinger operator with a short range potential, the point spectrum is non-degenerate and located on the negative real axis, while:  $\sigma_{ac}(Q_{0,0}(\mathcal{V})) = [0, +\infty)$ . Then,  $w(\zeta)$  does not annihilates almost everywhere in the closed upper complex plane, with the only possible exceptions of a discrete subset of  $i\mathbb{R}_+^*$ . Next, consider  $\zeta = k \in \mathbb{R}$  and let  $w_0(k)$  be the Wronskian associated to  $\{\chi_+(\cdot, -k), \chi_-(\cdot, k)\}$ ; the behaviour of  $w(k)$  on the real axis follows by using the relations

$$\chi_+(\cdot, k) = \frac{1}{2ik} (w_0^*(k) \chi_-(\cdot, k) - w(k) \chi_-(\cdot, -k)), \quad (2.22)$$

$$\chi_-(\cdot, k) = \frac{1}{2ik} (w_0(k) \chi_+(\cdot, k) - w(k) \chi_+(\cdot, -k)), \quad (2.23)$$

(e.g. in [29], chp. 5), expressing the Jost's solutions  $\chi_{\pm}(\cdot, k)$  in terms of the linear independent couples  $\chi_-(\cdot, \pm k)$  and  $\chi_+(\cdot, \pm k)$  respectively. Plugging (2.23) into (2.22), indeed, it follows:  $|w(k)|^2 = 4k^2 + |w_0(k)|^2$ , which entails

$$|w(k)|^2 \geq 4k^2. \quad (2.24)$$

Let introduce the functions  $\mathcal{G}^z(x, y)$  and  $\mathcal{H}^z(x, y)$

$$\mathcal{G}^z(x, y) = \frac{1}{w(\zeta)} \begin{cases} \chi_+(x, \zeta)\chi_-(y, \zeta), & x \geq y, \\ \chi_-(x, \zeta)\chi_+(y, \zeta), & x < y, \end{cases} \quad z = \zeta^2, \quad (2.25)$$

and

$$\mathcal{H}^z(x, y) = -\frac{1}{w(\zeta)} \begin{cases} \chi_+(x, \zeta)\partial_1\chi_-(y, \zeta), & x \geq y, \\ \chi_-(x, \zeta)\partial_1\chi_+(y, \zeta), & x < y, \end{cases}, \quad z = \zeta^2, \quad (2.26)$$

Assume  $\zeta \in \mathbb{C}^+$  be such that  $w(\zeta) \neq 0$  and  $y \in \mathbb{R}$ ; from the equation (2.17) and the relations (2.19), it follows that the maps  $x \rightarrow \mathcal{G}^z(\cdot, y)$  and  $x \rightarrow \mathcal{H}^z(\cdot, y)$  are exponentially decreasing as  $|x - y| \rightarrow \infty$  (with a decreasing rate depending on  $\text{Im } \zeta$ ) and fulfill the boundary condition problems

$$\begin{cases} (-\partial_x^2 + \mathcal{V} - \zeta^2) \mathcal{G}^z(\cdot, y) = 0 & \text{in } \mathbb{R} / \{y\} \\ \mathcal{G}^z(y^+, y) = \mathcal{G}^z(y^-, y), & \partial_1 \mathcal{G}^z(y^+, y) - \partial_1 \mathcal{G}^z(y^-, y) = -1, \end{cases} \quad (2.27)$$

and

$$\begin{cases} (-\partial_x^2 + \mathcal{V} - \zeta^2) \mathcal{H}^z(\cdot, y) = 0 & \text{in } \mathbb{R} / \{y\} \\ \mathcal{H}^z(y^+, y) - \mathcal{H}^z(y^-, y) = 1, & \partial_1 \mathcal{H}^z(y^+, y) = \partial_1 \mathcal{H}^z(y^-, y), \end{cases} \quad (2.28)$$

For  $z = \zeta^2$  s.t.  $w(\zeta) \neq 0$  and  $y \in \{a, b\}$ , the functions  $\mathcal{G}^z(\cdot, y)$ ,  $\mathcal{H}^z(\cdot, y)$  form a basis of the defect space  $\mathcal{N}_z$ , which writes as

$$\mathcal{N}_z = \text{l.c.} \{ \mathcal{G}^z(x, b), \mathcal{H}^z(x, b), \mathcal{G}^z(x, a), \mathcal{H}^z(x, a) \}. \quad (2.29)$$

According to the equation (2.27),  $\mathcal{G}^z(\cdot, y)$  identifies with the integral kernel of  $(Q_{0,0}(\mathcal{V}) - z)^{-1}$ , while, as a consequence of the definitions (2.25)-(2.26) and the results of the Proposition 2.1, the maps  $z \rightarrow \mathcal{G}^z(x, y)$ ,  $z \rightarrow \mathcal{H}^z(x, y)$  are meromorphic in  $\mathbb{C} \setminus \mathbb{R}_+$  with a branch cut along the positive real axis and poles, corresponding to the points in  $\sigma_p(Q_{0,0}(\mathcal{V}))$ , located on the negative real axis. In particular, due to the inequality (2.24), these functions continuously extend up to the branch cut, both in the limits:  $z \rightarrow k^2 \pm i0$ , with the only possible exception of the point  $z = 0$ .

In the case of positive defined potentials, it is possible to obtain uniform estimates of  $\mathcal{G}^z(x, y)$  and  $\mathcal{H}^z(x, y)$  up to the whole branch cut. Next, we assume  $\mathcal{V}$  to fulfill the additional condition

$$\langle u, \mathcal{V}u \rangle_{L^2(a,b)} > 0 \quad \forall u \in L^2(\mathbb{R}), \quad (2.30)$$

and introduce, for  $\zeta \in \mathbb{C}^+$  and  $z = \zeta^2$ , the functions

$$G^\zeta(x, y) = \mathcal{G}^{\zeta^2}(x, y); \quad \partial_1^i H^\zeta(x, y) = \partial_1^i \mathcal{H}^{\zeta^2}(x, y), \quad (2.31)$$

where the notation  $\partial^0 u = u$  is adopted. These are characterized as follows.

**Lemma 2.2** *Let  $\mathcal{V}$  fulfill (1.9) and (2.30). For all  $(x, y) \in \mathbb{R}^2$ ,  $x \neq y$ , the maps  $\zeta \rightarrow G^\zeta(x, y)$  and  $\zeta \rightarrow \partial_1^i H^\zeta(x, y)$ ,  $i = 0, 1$ , defined according to (2.31) are holomorphic in  $\mathbb{C}^+$  and continuously extend to  $\overline{\mathbb{C}^+}$ . In particular, for  $\zeta = k \in \mathbb{R}$ , it results:  $G^k(\cdot, y), H^k(\cdot, y) \in \mathcal{C}_x^1(\mathbb{R} \setminus \{y\})$ ,  $\mathcal{C}_k^0(\mathbb{R})$ , while the relations*

$$G^\zeta(x, y) = e^{i\zeta|x-y|} \mathcal{O}\left(\frac{1}{1+|\zeta|}\right), \quad H^\zeta(x, y) = e^{i\zeta|x-y|} \mathcal{O}(1), \quad \partial_1 H^\zeta(x, y) = e^{i\zeta|x-y|} \mathcal{O}(1+\zeta). \quad (2.32)$$

hold with  $\mathcal{O}(\cdot)$  referred to the metric space  $\mathbb{R}^2 \times \overline{\mathbb{C}^+}$ .

**Proof.** The conditions (1.9), (2.30) and the relation (2.24) prevent  $w(\zeta)$  to have zeroes in  $\overline{\mathbb{C}^+} \setminus \{0\}$ . Computing  $w(\zeta)$ , we have

$$w(\zeta) = \chi_+(a, \zeta)\partial_1\chi_-(a, \zeta) - \partial_1\chi_+(a, \zeta)\chi_-(a, \zeta). \quad (2.33)$$

Using the exterior conditions (2.18), the coefficients  $\partial_1^j \chi_-(a, \zeta)$ ,  $j = 0, 1$ , are explicetely given by

$$\chi_-(a, \zeta) = e^{-i\zeta a}, \quad \partial_1 \chi_-(a, \zeta) = -i\zeta e^{-i\zeta a}, \quad (2.34)$$

while, for  $x < b$ , the function  $\chi_+(\cdot, \zeta)$  writes as

$$\chi_+(x, \zeta) = e^{i\zeta x} b_+(x, \zeta), \quad (2.35)$$



where  $b_+(\cdot, \zeta)$  solves the equation

$$b_+(x, \zeta) = 1 - \int_x^b \mathcal{K}_+(t, x, \zeta) \mathcal{V}(t) b_+(t, \zeta) dt, \quad x < b, \quad (2.36)$$

with the integral kernel

$$\mathcal{K}_+(t, x, \zeta) = -e^{i\zeta(t-x)} \frac{\sin \zeta(t-x)}{\zeta}. \quad (2.37)$$

The solution of this problem represents in the form of a Picard's series  $b_+ = \sum_{n=0}^{+\infty} b_{+,n}$  whose terms are defined according to

$$b_{+,0} = 1, \quad b_{+,n}(x, \zeta) = - \int_x^b \mathcal{K}_+(t, x, \zeta) \mathcal{V}(t) b_{+,n-1}(t, \zeta) dt, \quad n \in \mathbb{N}^*, \quad x < b. \quad (2.38)$$

Under our assumptions (i.e. condition (1.9)), this series converges uniformly w.r.t.  $x \in \mathbb{R}$ ,  $\zeta \in \overline{\mathbb{C}^+}$ ; in particular, it results:  $\partial_1^j b_+(\cdot, \zeta) = \mathcal{O}(1)$ ,  $j = 0, 1$  (for this point, we refer to [29], chp. 5). Let  $\zeta = 0$ ; the equations (2.38) write as

$$b_{+,n}(x, 0) = \int_x^b (t-x) \mathcal{V}(t) b_{+,n-1}(t, 0) dt, \quad n \in \mathbb{N}^*, \quad x < b. \quad (2.39)$$

Using the conditions:  $\langle u, \mathcal{V}u \rangle_{L^2(a,b)} > 0$  and  $b_{+,0} = 1$ , an induction argument leads to:  $b_{+,n}(x, 0) \geq 0$  and  $b_+(x, 0) \geq 1$ . From (2.37) it follows

$$\partial_x b_+(x, 0) = - \int_x^b \mathcal{V}(t) b_+(t, 0) dt, \quad x < b. \quad (2.40)$$

Since  $b_+ > 0$  and  $\mathcal{V} > 0$ , at least in a subset of  $(a, b)$ , we have:  $\partial_1 b_+(a, 0) < 0$ . With the notation introduced above, the equation (2.33) rephrases as

$$w(\zeta) = -\partial_1 b_+(a, \zeta) - 2i\zeta b_+(a, \zeta). \quad (2.41)$$

Then, according to the conditions:  $\partial_1 b_+(a, 0) \neq 0$ , and  $b_+(\cdot, \zeta) = \mathcal{O}(1)$ , we have:  $w(\zeta) = -\partial_1 b_+(a, \zeta) + \mathcal{O}(\zeta)$  which implies  $w(0) \neq 0$ .

As a consequence, the function  $p(\zeta)$  defined by

$$p(\zeta) = \frac{1 + \zeta}{w(\zeta)}, \quad (2.42)$$

is bounded in any bounded set  $\mathcal{B}_R(0) \cap \overline{\mathbb{C}^+}$ ,  $R > 0$ . Moreover, using the equation (2.37), we have

$$b_+(a, \zeta) = 1 + \int_a^b \frac{e^{2i\zeta(t-x)} - 1}{\zeta} \mathcal{V}(t) b_+(t, \zeta) dt; \quad (2.43)$$

since  $b_+(\cdot, \zeta)$  is uniformly bounded for  $\zeta \in \overline{\mathbb{C}^+}$ , and  $|1/\zeta| |e^{2i\zeta(t-x)} - 1| \leq 2/|\zeta|$ , it follows:  $\lim_{\zeta \rightarrow \infty, \zeta \in \overline{\mathbb{C}^+}} b_+(a, \zeta) = 1$ . Set  $M = \sup_{\zeta \in \overline{\mathbb{C}^+}} |\partial_1 b_+(a, \zeta)|$ , and let  $\tilde{R} > 0$  be such that  $|\zeta| |b_+(a, \zeta)|/M > 1$  for any  $\zeta \in \overline{\mathbb{C}^+} \setminus \mathcal{B}_{\tilde{R}}(0)$ . From the representation (2.41) it follows

$$\sup_{\zeta \in \overline{\mathbb{C}^+} \setminus \mathcal{B}_{\tilde{R}}(0)} |p(\zeta)| \leq \frac{1}{M} \frac{1 + |\zeta|}{2|\zeta| \frac{|b_+(a, \zeta)|}{M} - 1} \lesssim 1. \quad (2.44)$$

Then,  $p(\zeta)$  results uniformly bounded as  $\zeta \in \overline{\mathbb{C}^+}$  and we can write

$$(w(\zeta))^{-1} = \mathcal{O}\left(\frac{1}{1 + |\zeta|}\right), \quad (2.45)$$

in the sense of the metric space  $\overline{\mathbb{C}^+}$  (see Definition 1.1).

From the definitions (2.25)-(2.26), (2.31) and the result of Proposition 2.1, the functions  $\zeta \rightarrow G^\zeta(x, y)$  and  $\zeta \rightarrow \partial_1^i H^\zeta(x, y)$ ,  $i = 0, 1$ , are meromorphic in  $\mathbb{C}^+$ , while, the previous result implies that, in our assumptions, these maps have no poles in  $\mathbb{C}^+$  and continuously extend to the whole real axis. The relations (2.32) follows from (2.19) by using (2.45). ■

## 2.2 Resolvent analysis.

The result of the previous Sections and, in particular, the Krein's-like formula given in (2.15), allow a detailed resolvent analysis for the operators  $Q_{\theta_1, \theta_2}(\mathcal{V})$ . At this concern, we recall that the maps  $z \rightarrow q(z, \mathcal{V})$  and  $z \rightarrow \gamma(e_i, z, \mathcal{V})$ , appearing at the r.h.s. of (2.15), are holomorphic in  $\mathbb{C} \setminus \sigma(Q_{0,0}(\mathcal{V}))$ , while, from the definitions (2.12)-(2.12), the matrix coefficients in  $A_{\theta_1, \theta_2}$  and  $B_{\theta_1, \theta_2}$  are analytic functions of the parameters  $(\theta_1, \theta_2)$  in the whole  $\mathbb{C}^2$ . Then

$$d(z, \theta_1, \theta_2) = \det(B_{\theta_1, \theta_2} q(z, \mathcal{V}) - A_{\theta_1, \theta_2}), \quad (2.46)$$

defines a separately holomorphic function of the variables  $z$  and  $(\theta_1, \theta_2)$ . Moreover, for any couple  $(\theta_1, \theta_2)$ , the set of singular points

$$\mathcal{S}_{\theta_1, \theta_2} = \{z \in \mathbb{C} \mid d(z, \theta_1, \theta_2) = 0\}, \quad (2.47)$$

is discrete. As a consequence, the representation (2.15) makes sense in the dense open set  $\mathbb{C} \setminus (\sigma(Q_{0,0}(\mathcal{V})) \cup \mathcal{S}_{\theta_1, \theta_2})$ . Let us fix  $z \in \mathbb{C} \setminus (\sigma(Q_{0,0}(\mathcal{V})) \cup \mathcal{S}_{\tilde{\theta}_1, \tilde{\theta}_2})$ , for a given couple  $(\tilde{\theta}_1, \tilde{\theta}_2) \in \mathbb{C}^2$ ; using the expansion

$$d(z, \theta_1, \theta_2) = d(z, \tilde{\theta}_1, \tilde{\theta}_2) + \mathcal{O}(\theta_1 - \tilde{\theta}_1) + \mathcal{O}(\theta_2 - \tilde{\theta}_2), \quad (2.48)$$

it results:  $d(z, \theta_1, \theta_2) \neq 0$  for all  $(\theta_1, \theta_2)$  in a suitable neighbourhood of  $(\tilde{\theta}_1, \tilde{\theta}_2)$ . This implies that, for any couple of parameters  $(\tilde{\theta}_1, \tilde{\theta}_2)$ , there exists  $z \in \rho(Q_{\tilde{\theta}_1, \tilde{\theta}_2}(\mathcal{V}))$  and a positive constant  $\delta$ , possibly depending on  $(\tilde{\theta}_1, \tilde{\theta}_2)$ , such that:  $z \in \rho(Q_{\theta_1, \theta_2}(\mathcal{V}))$  for all  $(\theta_1, \theta_2) \in \mathcal{B}_\delta((\tilde{\theta}_1, \tilde{\theta}_2))$ . Next, for such a  $z$ , consider the map  $(\theta_1, \theta_2) \rightarrow (Q_{\theta_1, \theta_2}(\mathcal{V}) - z)^{-1}$  defined for  $(\theta_1, \theta_2) \in \mathcal{B}_\delta((\tilde{\theta}_1, \tilde{\theta}_2))$ . Since  $z \notin \mathcal{S}_{\theta_1, \theta_2}$ , the coefficients of the finite rank part at the r.h.s. of (2.15) are holomorphic w.r.t.  $(\theta_1, \theta_2)$  and  $(Q_{\theta_1, \theta_2}(\mathcal{V}) - z)^{-1}$  forms an analytic family in  $\mathcal{L}(L^2(\mathbb{R}))$ . Then,  $Q_{\theta_1, \theta_2}(\mathcal{V})$  is analytic in the sense of Kato, w.r.t. the parameters  $(\theta_1, \theta_2)$ .

As this result suggests, when  $(\theta_1, \theta_2)$  is close to the origin of  $\mathbb{C}^2$ , a part of the point spectrum  $\sigma_p(Q_{\theta_1, \theta_2}(\mathcal{V}))$  is formed by non-degenerate eigenvalues holomorphically dependent from  $(\theta_1, \theta_2)$  and converging, in the limit  $(\theta_1, \theta_2) \rightarrow (0, 0)$ , to the corresponding points of  $\sigma_p(Q_{0,0}(\mathcal{V}))$  (see the point (ii) in the next Proposition 2.4). As an aside we notice that, for generic compactly supported potentials, new spectral points (not converging to  $\sigma_p(Q_{0,0}(\mathcal{V}))$ ) may eventually arise in a complex neighbourhood of the origin, due to the interface conditions. Nevertheless, if the additional assumption of positive potentials (2.30) is adopted, it is possible to prove the identity  $\sigma(Q_{\theta_1, \theta_2}(\mathcal{V})) = \sigma(Q_{0,0}(\mathcal{V}))$  provided that  $\theta_{i=1,2}$  are small enough. To fix this point, we need appropriate estimates for the coefficients of the finite rank part in (2.15).

The relations (2.15) and (2.16) can be made explicit by computing the matrix representation of  $(B_{\theta_1, \theta_2} q(z, \mathcal{V}) - A_{\theta_1, \theta_2})$  w.r.t. the basis  $\{e_j\}_{j=1}^4$  and (2.29). Making use of the definition (2.6), a direct computation yields

$$\gamma(\cdot, z, \mathcal{V}) = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}, \quad \text{with: } \begin{cases} \gamma(e_1, z, \mathcal{V}) = \mathcal{G}^z(x, b); & \gamma(e_2, z, \mathcal{V}) = -\mathcal{H}^z(x, b); \\ \gamma(e_3, z, \mathcal{V}) = \mathcal{G}^z(x, a); & \gamma(e_4, z, \mathcal{V}) = -\mathcal{H}^z(x, a). \end{cases} \quad (2.49)$$

The matrix coefficients of  $q(z, \mathcal{V})$  are related to the boundary values of the functions  $\gamma(e_i, z, \mathcal{V})$ ,  $i = 1 \dots 4$ , as  $x \rightarrow b^\pm$  or  $x \rightarrow a^\pm$ . Using the definitions (2.25) and (2.26), it follows:  $\partial_1 \mathcal{G}^z(x, y) = \mathcal{H}^z(y, x)$ ; in particular, the boundary values at  $x \rightarrow y^\pm$  are related by

$$\partial_1 \mathcal{G}^z(y^\pm, y) = \mathcal{H}^z(y^\mp, y). \quad (2.50)$$

Using these relations and the boundary conditions in (2.27)-(2.28), a direct computation yields

$$q(z, \mathcal{V}) = \begin{pmatrix} \mathcal{G}^z(b, b) & \frac{1}{2} - \mathcal{H}^z(b^+, b) & \mathcal{G}^z(b, a) & -\mathcal{H}^z(b, a) \\ \mathcal{H}^z(b^+, b) - \frac{1}{2} & -\partial_1 \mathcal{H}^z(b, b) & \mathcal{H}^z(a, b) & -\partial_1 \mathcal{H}^z(b, a) \\ \mathcal{G}^z(a, b) & -\mathcal{H}^z(a, b) & \mathcal{G}^z(a, a) & -(\frac{1}{2} + \mathcal{H}^z(a^-, a)) \\ \mathcal{H}^z(b, a) & -\partial_1 \mathcal{H}^z(a, b) & \mathcal{H}^z(a^-, a) + \frac{1}{2} & -\partial_1 \mathcal{H}^z(a, a) \end{pmatrix}. \quad (2.51)$$

**Lemma 2.3** *Let the matrix  $(B_{\theta_1, \theta_2} q(z, \mathcal{V}) - A_{\theta_1, \theta_2})$  be defined according to the relations (2.6), (2.11)-(2.12) and assume  $\mathcal{V}$  to fulfill the conditions (1.9), (2.30). There exists  $\delta > 0$  such that, for all  $(\theta_1, \theta_2) \in \mathcal{B}_\delta((0, 0))$ ,  $(B_{\theta_1, \theta_2} q(z, \mathcal{V}) - A_{\theta_1, \theta_2})$  is invertible in  $z \in \mathbb{C}/\mathbb{R}_+$ . The coefficients of the inverse matrix are separately holomorphic w.r.t.  $(\theta_1, \theta_2)$  and  $z$ , with:  $(\theta_1, \theta_2) \in \mathcal{B}_\delta((0, 0))$ ,  $z \in \mathbb{C}/\mathbb{R}_+$ ; they have continuous extensions to the branch cut both in the limits  $z = k^2 + i\varepsilon$ ,  $\varepsilon \rightarrow 0^\pm$ .*

**Proof.** Using the notation introduced in (2.31), for  $z = \zeta^2$ ,  $\zeta \in \mathbb{C}^+$ , a direct computation leads to

$$(B_{\theta_1, \theta_2} q(z, \mathcal{V}) - A_{\theta_1, \theta_2}) = \begin{pmatrix} \beta(\theta_2) (H^\zeta(b^+, b) - \frac{1}{2}) & -\beta(\theta_2) \partial_1 H^\zeta(b, b) & \beta(\theta_2) H^\zeta(a, b) & -\beta(\theta_2) \partial_1 H^\zeta(b, a) \\ \beta(\theta_1) G^\zeta(b, b) & \beta(\theta_1) (\frac{1}{2} - H^\zeta(b^+, b)) & \beta(\theta_1) G^\zeta(b, a) & -\beta(\theta_1) H^\zeta(b, a) \\ \beta(-\theta_2) H^\zeta(b, a) & -\beta(-\theta_2) \partial_1 H^\zeta(a, b) & \beta(-\theta_2) (H^\zeta(a^-, a) + \frac{1}{2}) & -\beta(-\theta_2) \partial_1 H^\zeta(a, a) \\ \beta(-\theta_1) G^\zeta(a, b) & -\beta(-\theta_1) H^\zeta(a, b) & \beta(-\theta_1) G^\zeta(a, a) & -\beta(-\theta_1) (\frac{1}{2} + H^\zeta(a^-, a)) \end{pmatrix} - \begin{pmatrix} \alpha(\theta_2) & & & \\ & \alpha(\theta_1) & & \\ & & \alpha(-\theta_2) & \\ & & & \alpha(-\theta_1) \end{pmatrix} \quad (2.52)$$

where  $\alpha(\theta)$  and  $\beta(\theta)$  are defined by

$$\alpha(\theta) = 1 + e^{\frac{\theta}{2}}, \quad \beta(\theta) = 1 - e^{\frac{\theta}{2}}. \quad (2.53)$$

As consequence of the Lemma 2.2, for positive defined potentials the above relation rephrases as

$$(B_{\theta_1, \theta_2} q(z, \mathcal{V}) - A_{\theta_1, \theta_2}) = \begin{pmatrix} \beta(\theta_2) \mathcal{O}(1) - \alpha(\theta_2) & \beta(\theta_2) \mathcal{O}(1 + \zeta) & \beta(\theta_2) e^{i\zeta(b-a)} \mathcal{O}(1) & \beta(\theta_2) e^{i\zeta(b-a)} \mathcal{O}(1 + \zeta) \\ \beta(\theta_1) \mathcal{O}\left(\frac{1}{1+|\zeta|}\right) & \beta(\theta_1) \mathcal{O}(1) - \alpha(\theta_1) & \beta(\theta_1) e^{i\zeta(b-a)} \mathcal{O}\left(\frac{1}{1+|\zeta|}\right) & \beta(\theta_1) e^{i\zeta(b-a)} \mathcal{O}(1) \\ \beta(-\theta_2) e^{i\zeta(b-a)} \mathcal{O}(1) & \beta(-\theta_2) e^{i\zeta(b-a)} \mathcal{O}(1 + \zeta) & \beta(-\theta_2) \mathcal{O}(1) - \alpha(-\theta_2) & \beta(-\theta_2) \mathcal{O}(1 + \zeta) \\ \beta(-\theta_1) e^{i\zeta(b-a)} \mathcal{O}\left(\frac{1}{1+|\zeta|}\right) & \beta(-\theta_1) e^{i\zeta(b-a)} \mathcal{O}(1) & \beta(-\theta_1) \mathcal{O}\left(\frac{1}{1+|\zeta|}\right) & \beta(-\theta_1) \mathcal{O}(1) - \alpha(-\theta_1) \end{pmatrix}, \quad (2.54)$$

being the symbols  $\mathcal{O}(\cdot)$  referred to the metric space  $\overline{\mathbb{C}^+}$  and defining holomorphic functions of  $\zeta \in \mathbb{C}^+$  with continuous extension the real axis. Due to the definition of  $\alpha(\theta)$ ,  $\beta(\theta)$ , the coefficients of  $(B_{\theta_1, \theta_2} q(z, \mathcal{V}) - A_{\theta_1, \theta_2})$  result separately holomorphic w.r.t.  $(\theta_1, \theta_2) \in \mathbb{C}^2$ ,  $z \in \mathbb{C}/\mathbb{R}_+$ , and admit, for each couple  $(\theta_1, \theta_2)$ , continuous extensions to the branch cut. In particular, setting  $\zeta = k \in \mathbb{R}_\pm$  at the r.h.s. of (2.54) correspond to consider the limits of  $(B_{\theta_1, \theta_2} q(z, \mathcal{V}) - A_{\theta_1, \theta_2})_{ij}$  for  $z \rightarrow k^2 \pm i0$  respectively. Making use of this expression, and taking into account the definitions (2.53), a determinant's expansion follows

$$d(z, \theta_1, \theta_2) = 4 \left( 1 + \cosh \frac{\theta_1}{2} \right) \left( 1 + \cosh \frac{\theta_2}{2} \right) + \mathcal{O}(\theta_1) + \mathcal{O}(\theta_2), \quad (2.55)$$

where  $\mathcal{O}(\theta_i)$ , being referred to the metric space  $\mathcal{B}_1((0, 0)) \times \mathbb{C}$ , defines a separately holomorphic functions w.r.t.  $(\theta_1, \theta_2) \in \mathcal{B}_1((0, 0))$  and  $z \in \mathbb{C}/\mathbb{R}_+$ , allowing continuous extensions to the branch cut in the above-specified sense. According to the Definition 1.1,  $\mathcal{O}(\theta_i)$  writes as

$$\mathcal{O}(\theta_i) = \theta_i p(z, \theta_1, \theta_2), \quad (2.56)$$

with  $p(z, \theta_1, \theta_2)$  uniformly bounded in  $\mathcal{B}_1((0, 0)) \times \mathbb{C}$ . Therefore  $\delta > 0$  exists such that, when  $(\theta_1, \theta_2) \in \mathcal{B}_\delta((0, 0))$ , it results  $|d(z, \theta_1, \theta_2)| > 1$  for all  $z \in \mathbb{C}/\mathbb{R}_+$ . In these conditions, the matrix  $(B_{\theta_1, \theta_2} q(z, \mathcal{V}) - A_{\theta_1, \theta_2})$  is invertible and the coefficients  $(B_{\theta_1, \theta_2} q(z, \mathcal{V}) - A_{\theta_1, \theta_2})_{ij}^{-1}$  are separately holomorphic w.r.t.  $(\theta_1, \theta_2) \in \mathcal{B}_\delta((0, 0))$ ,  $z \in \mathbb{C}/\mathbb{R}_+$ , having continuous extensions to the whole branch cut, both in the limits  $z = k^2 + i\varepsilon$ ,  $\varepsilon \rightarrow 0^\pm$ . ■

We are now in the position to develop the spectral analysis for the operators  $Q_{\theta_1, \theta_2}(\mathcal{V})$  starting from the resolvent's formula (2.15).

**Proposition 2.4** *Let  $Q_{\theta_1, \theta_2}(\mathcal{V})$  be defined according to (1.9), (1.11). The operator's spectrum characterizes as follows:*

- i) *For any  $(\theta_1, \theta_2) \in \mathbb{C}^2$ , the essential part of the spectrum is  $\sigma_{ess}(Q_{\theta_1, \theta_2}(\mathcal{V})) = \mathbb{R}_+$ .*
- ii) *Let  $E_0$  be an eigenvalue of  $Q_{0,0}(\mathcal{V})$  and assume  $\varepsilon_0 > 0$  small enough; for any fixed  $\varepsilon \in (0, \varepsilon_0)$  it exists  $\delta_\varepsilon > 0$  depending on  $\varepsilon$  s.t.: for any  $(\theta_1, \theta_2) \in \mathcal{B}_{\delta_\varepsilon}((0, 0)) \cap \mathbb{C}^2$  it exists an unique nondegenerate and discrete eigenvalue  $E(\theta_1, \theta_2) \in \sigma(Q_{\theta_1, \theta_2}(\mathcal{V})) \cap \mathcal{B}_\varepsilon(E_0)$ . Moreover, the function  $E(\theta_1, \theta_2)$  is holomorphic w.r.t.  $(\theta_1, \theta_2)$  in  $\mathcal{B}_{\delta_\varepsilon}((0, 0))$ .*
- If, in addition,  $\mathcal{V}$  is assumed to be positive defined, fulfilling (2.30), then:
- iii) *It exists  $\delta > 0$  s.t., for all  $(\theta_1, \theta_2) \in \mathcal{B}_\delta((0, 0))$ ,  $\sigma(Q_{\theta_1, \theta_2}(\mathcal{V}))$  is purely absolutely continuous and coincide with the positive real axis.*

**Proof.** With the notation introduced above, let  $z \in \mathbb{C} \setminus (\sigma(Q_{0,0}(\mathcal{V})) \cup \mathcal{S}_{\theta_1, \theta_2})$ ; the representation (2.15) implies that the difference  $(Q_{\theta_1, \theta_2}(\mathcal{V}) - z)^{-1} - (Q_{0,0}(\mathcal{V}) - z)^{-1}$  is a finite rank operator. Then, the first statement of the Proposition follows by adapting the Weyl's theorem to the non-selfadjoint framework (for this point, we refer to [25], Sec. XIII.4, Lemma 3 and the strong spectral mapping theorem).

In the unperturbed case,  $Q_{0,0}(\mathcal{V})$  is a 1D Schrödinger operator with a short range potential. Its spectrum has a purely absolutely continuous part on the positive real axis, and possible non-degenerate eigenvalues located on the negative real axis, without accumulation points. Then, the second statement is a direct consequence of the Kato-Rellich theorem, since  $Q_{\theta_1, \theta_2}(\mathcal{V})$  is Kato-analytic w.r.t. the parameters.

When  $\mathcal{V}$  is a positive defined potential,  $\sigma(Q_{0,0}(\mathcal{V})) = \sigma_{ac}(Q_{0,0}(\mathcal{V})) = \mathbb{R}_+$ , while the point spectrum is empty. The spectrum  $\sigma(Q_{\theta_1, \theta_2}(\mathcal{V}))$  corresponds to the subset of the complex plane where the map  $z \rightarrow \mathcal{G}_{\theta_1, \theta_2}^z(x, y)$  (defined in eq. (2.16)) is not holomorphic. According to the result of the Lemma 2.2, the functions  $\mathcal{G}^z(x, y)$  and  $\mathcal{H}^z(x, y)$ , appearing at the r.h.s. of (2.16) are  $z$ -holomorphic in  $\mathbb{C}/\mathbb{R}_+$  and continuously extend to the whole branch cut both in the limits  $z = k^2 + i\varepsilon$ ,  $\varepsilon \rightarrow 0^\pm$ . As shown in Lemma 2.3, the same hold for the coefficients of  $(B_{\theta_1, \theta_2} q(z, \mathcal{V}) - A_{\theta_1, \theta_2})^{-1}$ , provided that  $(\theta_1, \theta_2)$  is close enough to the origin in  $\mathbb{C}^2$ . In particular, these result  $z$ -holomorphic in  $\mathbb{C}/\mathbb{R}_+$  and have continuous extensions to the whole branch cut. Thus, for  $\mathcal{V}$  positive defined, the map  $z \rightarrow \mathcal{G}_{\theta_1, \theta_2}^z(x, y)$  is holomorphic in  $\mathbb{C}/\mathbb{R}_+$  and have continuous extensions as  $z \rightarrow \mathbb{R}_+$ , both in the limits  $z = k^2 + i\varepsilon$ ,  $\varepsilon \rightarrow 0^\pm$ . This yields:  $\sigma(Q_{\theta_1, \theta_2}(\mathcal{V})) = \sigma_{ac}(Q_{\theta_1, \theta_2}(\mathcal{V})) = \mathbb{R}_+$ . ■

### 2.3 Generalized eigenfunctions expansion.

Let  $\psi_-(\cdot, k, \theta_1, \theta_2)$  denotes the generalized eigenfunction of the operator  $Q_{\theta_1, \theta_2}(\mathcal{V})$ , describing an incoming wave function of momentum  $k$ ; this is a solutions of the boundary value problem

$$\begin{cases} (-\partial_x^2 + \mathcal{V}) u = k^2 u, & \text{for } x \in \mathbb{R} \setminus \{a, b\}, \quad k \in \mathbb{R}, \\ e^{-\frac{\theta_1}{2}} u(b^+, \zeta, \theta_1, \theta_2) = u(b^-, \zeta, \theta_1, \theta_2), & e^{-\frac{\theta_2}{2}} u'(b^+, \zeta, \theta_1, \theta_2) = u'(b^-, \zeta, \theta_1, \theta_2), \\ e^{-\frac{\theta_1}{2}} u(a^-, \zeta, \theta_1, \theta_2) = u(a^+, \zeta, \theta_1, \theta_2), & e^{-\frac{\theta_2}{2}} u'(a^-, \zeta, \theta_1, \theta_2) = u'(a^+, \zeta, \theta_1, \theta_2), \end{cases} \quad (2.57)$$

fulfilling the exterior conditions

$$\psi_-(x, k, \theta_1, \theta_2) \Big|_{\substack{x < a \\ k > 0}} = e^{ikx} + R(k, \theta_1, \theta_2) e^{-ikx}, \quad \psi_-(x, k, \theta_1, \theta_2) \Big|_{\substack{x > b \\ k > 0}} = T(k, \theta_1, \theta_2) e^{ikx}, \quad (2.58)$$

and

$$\psi_-(x, k, \theta_1, \theta_2) \Big|_{\substack{x < a \\ k < 0}} = T(k, \theta_1, \theta_2) e^{ikx}, \quad \psi_-(x, k, \theta_1, \theta_2) \Big|_{\substack{x > b \\ k < 0}} = e^{ikx} + R(k, \theta_1, \theta_2) e^{-ikx}, \quad (2.59)$$

where  $R$  and  $T$  are the reflection and transmission coefficients. In the case  $(\theta_1, \theta_2) = (0, 0)$ ,  $\psi_-(\cdot, k, 0, 0)$  is a generalized eigenfunction of the selfadjoint model  $Q_{0,0}(\mathcal{V})$ : In what follows we adopt the simplified notation:  $\psi_-(\cdot, k, 0, 0) = \psi_-(\cdot, k)$ . These functions expresses in terms of the corresponding the Jost's solutions associated to  $Q_{0,0}(\mathcal{V})$  as

$$\psi_-(x, k) = \begin{cases} -\frac{2ik}{w(k)} \chi_+(x, k), & \text{for } k \geq 0, \\ \frac{2ik}{w(-k)} \chi_-(x, -k), & \text{for } k < 0, \end{cases} \quad (2.60)$$

(e.g. in [29]). In the case of positive defined potentials, an approach similar to the one leading to the Krein-like resolvent formula (2.9) allows to obtain an expansion for the difference:  $\psi_-(\cdot, k, \theta_1, \theta_2) - \psi_-(\cdot, k)$  as  $(\theta_1, \theta_2) \rightarrow (0, 0)$ . To this aim, we need an explicit expression of the finite rank terms, appearing at the r.h.s. of (2.15), in the limits where  $z$  approaches the branch cut. This can be done by using the results of Lemmas 2.2 and 2.3. Adopting the notation introduced in (2.31), let us define

$$\{g(e_i, \zeta, \mathcal{V})\}_{i=1}^4 = \{G^\zeta(\cdot, b), -H^\zeta(\cdot, b), G^\zeta(\cdot, a), -H^\zeta(\cdot, a)\}; \quad (2.61)$$

we get, for  $\zeta \in \mathbb{C}^+$  and  $z = \zeta^2$ , the identity:  $\gamma(e_i, z, \mathcal{V}) = g(e_i, \zeta, \mathcal{V})$ ; due to Lemma 2.2, the limits of  $g(e_i, \zeta, \mathcal{V})$  as  $\zeta \rightarrow k \in \mathbb{R}_\pm$  exist and corresponds to the limits of  $\gamma(e_i, z, \mathcal{V})$  as  $z \rightarrow k^2 \pm i0$  respectively. Namely, we have

$$\lim_{z \rightarrow k^2 \pm i0} \gamma(e_i, z, \mathcal{V}) = \begin{cases} g(e_i, k, \mathcal{V})|_{k \in \mathbb{R}_+} \\ g(e_i, k, \mathcal{V})|_{k \in \mathbb{R}_-} \end{cases}. \quad (2.62)$$

The coefficients  $(B_{\theta_1, \theta_2} q(z, \mathcal{V}) - A_{\theta_1, \theta_2})_{ij}^{-1}$  have been considered in the Lemma 2.3 where their regularity w.r.t. the  $z$  and the extensions to the branch cut have been investigated. To get further insights on the structure of

the inverse matrix, we use the explicit form of  $(B_{\theta_1, \theta_2} q(z, \mathcal{V}) - A_{\theta_1, \theta_2})$  given in (2.52)-(2.54). In what follows,  $\mathcal{M}(\zeta, \theta_1, \theta_2)$  denotes the r.h.s. of (2.52)

$$(B_{\theta_1, \theta_2} q(z, \mathcal{V}) - A_{\theta_1, \theta_2}) = \mathcal{M}(\zeta, \theta_1, \theta_2), \quad \zeta \in \mathbb{C}^+, \quad z = \zeta^2. \quad (2.63)$$

In the assumption (2.30), the matrix  $\mathcal{M}(\zeta, \theta_1, \theta_2)$  continuously extends to  $\zeta \in \overline{\mathbb{C}^+}$  and taking its limits for  $\zeta \rightarrow k \in \mathbb{R}_\pm$  corresponds to consider the limits of  $(B_{\theta_1, \theta_2} q(z, \mathcal{V}) - A_{\theta_1, \theta_2})_{ij}$  as  $z \rightarrow k^2 \pm i0$  respectively. This yields

$$\lim_{z \rightarrow k^2 \pm i0} (B_{\theta_1, \theta_2} q(z, \mathcal{V}) - A_{\theta_1, \theta_2}) = \begin{cases} \mathcal{M}(k, \theta_1, \theta_2)|_{k \in \mathbb{R}_+^*} \\ \mathcal{M}(k, \theta_1, \theta_2)|_{k \in \mathbb{R}_-} \end{cases}. \quad (2.64)$$

In particular, making use of (2.54), we have

$$\mathcal{M}(k, \theta_1, \theta_2) = \begin{pmatrix} \beta(\theta_2) \mathcal{O}(1) - \alpha(\theta_2) & \beta(\theta_2) \mathcal{O}(1+k) & \beta(\theta_2) \mathcal{O}(1) & \beta(\theta_2) \mathcal{O}(1+k) \\ \beta(\theta_1) \mathcal{O}\left(\frac{1}{1+|k|}\right) & \beta(\theta_1) \mathcal{O}(1) - \alpha(\theta_1) & \beta(\theta_1) \mathcal{O}\left(\frac{1}{1+|k|}\right) & \beta(\theta_1) \mathcal{O}(1) \\ \beta(-\theta_2) \mathcal{O}(1) & \beta(-\theta_2) \mathcal{O}(1+k) & \beta(-\theta_2) \mathcal{O}(1) - \alpha(-\theta_2) & \beta(-\theta_2) \mathcal{O}(1+k) \\ \beta(-\theta_1) \mathcal{O}\left(\frac{1}{1+|k|}\right) & \beta(-\theta_1) \mathcal{O}(1) & \beta(-\theta_1) \mathcal{O}\left(\frac{1}{1+|k|}\right) & \beta(-\theta_1) \mathcal{O}(1) - \alpha(-\theta_1) \end{pmatrix}. \quad (2.65)$$

From the Lemma 2.3, this matrix is invertible whenever  $(\theta_1, \theta_2) \in \mathcal{B}_\delta((0, 0))$  with  $\delta$  small enough; under such a condition, indeed, the determinant's expansion

$$\det \mathcal{M}(k, \theta_1, \theta_2) = \det(-A_{\theta_1, \theta_2}) + \mathcal{O}(\theta_1) + \mathcal{O}(\theta_2),$$

$$\det(-A_{\theta_1, \theta_2}) = 4 \left(1 + \cosh \frac{\theta_1}{2}\right) \left(1 + \cosh \frac{\theta_2}{2}\right),$$

(see the relation (2.55)) implies:  $|\det \mathcal{M}(k, \theta_1, \theta_2)| > 1$ . Then, a direct computation leads to

$$\mathcal{M}^{-1}(k, \theta_1, \theta_2) = \frac{1}{\det \mathcal{M}(k, \theta_1, \theta_2)} [\det(A_{\theta_1, \theta_2}) \text{diag}\{\lambda_i\} + M(k, \theta_1, \theta_2)], \quad (2.66)$$

where  $\text{diag}(\lambda_i)$ , the main term in (2.66), is the  $\mathbb{C}^{4,4}$  diagonal matrix defined by the coefficients

$$\{\lambda_i\}_{i=1}^4 = \left\{ \frac{-1}{\alpha(\theta_2)}, \frac{-1}{\alpha(\theta_1)}, \frac{-1}{\alpha(-\theta_2)}, \frac{-1}{\alpha(-\theta_1)} \right\}, \quad (2.67)$$

while the remainder is

$$M(k, \theta_1, \theta_2) = \begin{pmatrix} \mathcal{O}(\theta_1) + \mathcal{O}(\theta_2) & \mathcal{O}(\theta_2(1+k)) & \mathcal{O}(\theta_2) & \mathcal{O}(\theta_2(1+k)) \\ \mathcal{O}\left(\frac{\theta_1}{1+|k|}\right) & \mathcal{O}(\theta_1) + \mathcal{O}(\theta_2) & \mathcal{O}\left(\frac{\theta_1}{1+|k|}\right) & \mathcal{O}(\theta_1) \\ \mathcal{O}(\theta_2) & \mathcal{O}(\theta_2(1+k)) & \mathcal{O}(\theta_1) + \mathcal{O}(\theta_2) & \mathcal{O}(\theta_2(1+k)) \\ \mathcal{O}\left(\frac{\theta_1}{1+|k|}\right) & \mathcal{O}(\theta_1) & \mathcal{O}\left(\frac{\theta_1}{1+|k|}\right) & \mathcal{O}(\theta_1) + \mathcal{O}(\theta_2) \end{pmatrix}, \quad (2.68)$$

Here, the symbols  $\mathcal{O}(\cdot)$  are referred to the metric space  $\mathcal{B}_\delta((0, 0)) \times \mathbb{R}$  and, being obtained from the calculus of the inverse matrix  $(B_{\theta_1, \theta_2} q(z, \mathcal{V}) - A_{\theta_1, \theta_2})^{-1}$ , denotes polynomial expressions depending on the functions:  $\alpha(\pm\theta_i)$ ,  $\beta(\pm\theta_i)$ ,  $G^\zeta(x, y)$ ,  $\partial_1^i H^\zeta(x, y)$ , with  $x, y \in \{a, b\}$  and  $i = 0, 1$ . Then, as a consequence of Lemma 2.2, these terms are holomorphic w.r.t. the parameters  $(\theta_1, \theta_2) \in \mathcal{B}_\delta((0, 0))$  and continuous w.r.t.  $k \in \mathbb{R}$ .

**Proposition 2.5** Assume  $(\theta_1, \theta_2) \in \mathcal{B}_\delta((0, 0))$  with  $\delta > 0$  small enough, and let  $\mathcal{V}$  be defined according to (1.9), (2.30). The solutions  $\psi_-(\cdot, k, \theta_1, \theta_2)$  of the generalized eigenfunctions problem (2.57), (2.58)-(2.59) allow the representation

$$\psi_-(\cdot, k, \theta_1, \theta_2) = \begin{cases} \psi_-(\cdot, k) - \sum_{i,j=1}^4 [\mathcal{M}^{-1}(k, \theta_1, \theta_2) B_{\theta_1, \theta_2}]_{ij} [\Gamma_2 \psi_-(\cdot, k)]_j g(e_i, k, \mathcal{V}), & \text{for } k \geq 0, \\ \psi_-(\cdot, k) - \sum_{i,j=1}^4 [\mathcal{M}^{-1}(-k, \theta_1, \theta_2) B_{\theta_1, \theta_2}]_{ij} [\Gamma_2 \psi_-(\cdot, k)]_j g(e_i, -k, \mathcal{V}), & \text{for } k < 0. \end{cases} \quad (2.69)$$

The functions  $\psi_-(x, k, \theta_1, \theta_2)$  are  $\mathcal{C}^1$ -continuous w.r.t.  $x \in \mathbb{R} \setminus \{a, b\}$ ,  $k$ -continuous in  $\mathbb{R}$  and holomorphic w.r.t. the parameters  $(\theta_1, \theta_2)$  in  $\mathcal{B}_\delta((0, 0))$ .

**Proof.** We start considering the case  $k \geq 0$ . According to the definition of  $\psi_-(\cdot, k)$  and  $g(e_i, k, \mathcal{V})$ , the function at the r.h.s. of (2.69) solves the equation

$$(-\partial_x^2 + \mathcal{V})u = k^2 u, \quad \text{for } x \in \mathbb{R} \setminus \{a, b\}, \quad k \in \mathbb{R}, \quad (2.70)$$

and fulfills the conditions (2.58) and (2.59). Set:  $\psi_-(\cdot, k, \theta_1, \theta_2) = \phi - \psi$  with

$$\phi = \psi_-(\cdot, k), \quad (2.71)$$

$$\psi = \sum_{i,j=1}^4 [\mathcal{M}^{-1}(k, \theta_1, \theta_2) B_{\theta_1, \theta_2}]_{ij} [\Gamma_2 \phi]_j g(e_i, k, \mathcal{V}). \quad (2.72)$$

The function  $\psi$  can be pointwise approximated by elements of the defect spaces  $\mathcal{N}_z$  as  $z \rightarrow k^2 + i0$ . With the notation introduced in (2.63) and (2.61), let  $\psi_z$  be defined by

$$\psi_z = \sum_{i,j=1}^4 [\mathcal{M}^{-1}(\zeta, \theta_1, \theta_2) B_{\theta_1, \theta_2}]_{ij} [\Gamma_2 \phi]_j g(e_i, \zeta, \mathcal{V}), \quad \zeta \in \mathbb{C}^+, \quad z = \zeta^2; \quad (2.73)$$

it results  $\psi_z \in \mathcal{N}_z$  and:  $\lim_{z \rightarrow k^2 + i0} \psi_z = \psi$ . Since  $\psi_-(\cdot, k)$  is  $\mathcal{C}_x^1$ -continuous in  $\mathbb{R}$ , it results:  $\Gamma_1 \phi = 0$  and the following relation holds

$$\begin{aligned} \mathcal{M}(k, \theta_1, \theta_2) \Gamma_1 (\phi - \psi) &= -\mathcal{M}(k, \theta_1, \theta_2) \Gamma_1 \psi = -\lim_{z \rightarrow k^2 + i0} (B_{\theta_1, \theta_2} q(z, \mathcal{V}) - A_{\theta_1, \theta_2}) \Gamma_1 \psi_z \\ &= -\lim_{z \rightarrow k^2 + i0} (B_{\theta_1, \theta_2} \Gamma_2 \gamma(\cdot, z, \mathcal{V}) - A_{\theta_1, \theta_2}) \Gamma_1 \psi_z = -\lim_{z \rightarrow k^2 + i0} (B_{\theta_1, \theta_2} \Gamma_2 - A_{\theta_1, \theta_2} \Gamma_1) \psi_z \\ &= (-B_{\theta_1, \theta_2} \Gamma_2 + A_{\theta_1, \theta_2} \Gamma_1) \psi. \end{aligned} \quad (2.74)$$

The  $n$ -th component of the vector at the l.h.s. of (2.74) writes as

$$\begin{aligned} [\mathcal{M}(k, \theta_1, \theta_2) \Gamma_1 (\phi - \psi)]_n &= [-\mathcal{M}(k, \theta_1, \theta_2) \Gamma_1 \psi]_n = \\ &= -\sum_{i,j=1}^4 [[\mathcal{M}(k, \theta_1, \theta_2) \Gamma_1 (\phi - \psi)]_n \Gamma_1 g(e_i, k, \mathcal{V})]_n [\mathcal{M}^{-1}(k, \theta_1, \theta_2) B_{\theta_1, \theta_2}]_{ij} [\Gamma_2 \phi]_j. \end{aligned}$$

Recalling that  $\Gamma_1 g(e_i, k, \mathcal{V}) = e_i$ , we get

$$\begin{aligned} [\mathcal{M}(k, \theta_1, \theta_2) \Gamma_1 (\phi - \psi)]_n &= \\ &= -\sum_{i,j=1}^4 (\mathcal{M}(k, \theta_1, \theta_2))_{ni} [\mathcal{M}^{-1}(k, \theta_1, \theta_2) B_{\theta_1, \theta_2}]_{ij} [\Gamma_2 \phi]_j = -\sum_{i,j=1}^4 B_{nj} [\Gamma_2 \phi]_j, \end{aligned}$$

which implies

$$\mathcal{M}(k, \theta_1, \theta_2) \Gamma_1 (\phi - \psi) = -B \Gamma_2 \phi. \quad (2.75)$$

From (2.74) and (2.75), the interface conditions

$$A_{\theta_1, \theta_2} \Gamma_1 \psi_-(\cdot, k, \theta_1, \theta_2) = B_{\theta_1, \theta_2} \Gamma_2 \psi_-(\cdot, k, \theta_1, \theta_2), \quad (2.76)$$

follows. Since these are equivalent to the ones assigned in the equation (2.57), the function defined in (2.69) is a solution of the problem (2.57), (2.58)-(2.59). The case  $k < 0$  can be treated by a suitable adaptation of the above arguments.

The regularity of these functions w.r.t. to the variables  $\{x, k, \theta_1, \theta_2\}$  is a consequence of the representation (2.69) and of the properties of the maps  $\psi_-(\cdot, k)$ ,  $\mathcal{M}^{-1}(k, \theta_1, \theta_2)$ ,  $B_{\theta_1, \theta_2}$ , and  $g(e_i, k, \mathcal{V})$  (for this point, we refer to the corresponding definitions and to the results of the Proposition 2.1 and of the Lemmata 2.2-2.3). ■

As a consequence of this result, an expansion of  $\psi_-(\cdot, k, \theta_1, \theta_2)$  for small values of  $\theta_i$  follows.

**Corollary 2.6** *Let  $\psi_-(\cdot, k, \theta_1, \theta_2)$  denotes a solution of the generalized eigenfunctions problem (2.57), (2.58)-(2.59). In the assumptions of the Proposition 2.5, the expansion*

$$\psi_-(\cdot, k, \theta_1, \theta_2) - \psi_-(\cdot, k) = \mathcal{O}(\theta_2 k) G^{\sigma k}(\cdot, b) + \mathcal{O}\left(\frac{\theta_1 k}{1 + |k|}\right) H^{\sigma k}(\cdot, b) + \mathcal{O}(\theta_2 k) G^{\sigma k}(\cdot, a) + \mathcal{O}\left(\frac{\theta_1 k}{1 + |k|}\right) H^{\sigma k}(\cdot, a). \quad (2.77)$$

holds with:  $\sigma = \frac{k}{|k|}$ . The symbols  $\mathcal{O}(\cdot)$ , defined in the sense of the metric space  $\mathbb{R} \times \mathcal{B}_\delta((0, 0))$ .

**Proof.** As already noticed, the assumption of positive potentials (2.30) prevents the Jost's function  $w(k)$  to have zeroes on the real axis. In particular, a consequence of the definition (2.60) and of the relations (2.19) is

$$\psi_-(x, k) = \mathcal{O}\left(\frac{k}{1 + |k|}\right); \quad \partial_x \psi_-(x, k) = \mathcal{O}(k), \quad (2.78)$$

and a direct computation yields

$$B_{\theta_1, \theta_2} \Gamma_2 \psi_-(\cdot, k) = \left\{ \mathcal{O}(\theta_2 k), \mathcal{O}\left(\frac{\theta_1 k}{1+|k|}\right), \mathcal{O}(\theta_2 k), \mathcal{O}\left(\frac{\theta_1 k}{1+|k|}\right) \right\}. \quad (2.79)$$

where the symbols  $\mathcal{O}(\cdot)$  are referred to the metric space  $\mathbb{R} \times \mathcal{B}_\delta((0, 0))$ . Making use of this expression and of the relations (2.66)-(2.68), we get

$$\mathcal{M}^{-1}(\sigma k, \theta_1, \theta_2) B_{\theta_1, \theta_2} [\Gamma_2 \psi_-(\cdot, k)] = \left\{ \mathcal{O}(\theta_2 k), \mathcal{O}\left(\frac{\theta_1 k}{1+|k|}\right), \mathcal{O}(\theta_2 k), \mathcal{O}\left(\frac{\theta_1 k}{1+|k|}\right) \right\}. \quad (2.80)$$

Then, the expansion (2.77) follows from the formula (2.69) by taking into account (2.80) and the definition (2.61). ■

### 3 Similarity and uniform-in-time estimates for the dynamical system.

In what follows,  $\mathcal{V}$  is a positive short-range potential. With this assumption,  $Q_{0,0}(\mathcal{V})$  has a purely absolutely continuous spectrum and the related generalized Fourier transform  $\mathcal{F}_\mathcal{V}$

$$(\mathcal{F}_\mathcal{V} \varphi)(k) = \int_{\mathbb{R}} \frac{dx}{(2\pi)^{1/2}} \psi_-^*(x, k) \varphi(x), \quad \varphi \in L^2(\mathbb{R}), \quad (3.1)$$

is a unitary map with range:  $R(\mathcal{F}_\mathcal{V}) = L^2(\mathbb{R})$  and  $\mathcal{F}_\mathcal{V}^{-1}$  acting as:  $(\mathcal{F}_\mathcal{V}^{-1} f)(x) = \int \frac{dk}{(2\pi)^{1/2}} \psi_-(x, k) f(k)$ , for all  $f \in L^2(\mathbb{R})$ . Assume in addition the parameters  $\theta_1, \theta_2$  close enough to the origin, so that expansion (2.77) hold, and consider the operator  $\mathcal{W}_{\theta_1, \theta_2}$  defined by the integral kernel

$$\mathcal{W}_{\theta_1, \theta_2}(x, y) = \int_{\mathbb{R}} \frac{dk}{2\pi} \psi_-(x, k, \theta_1, \theta_2) \psi_-^*(y, k). \quad (3.2)$$

The next Proposition shows that  $\mathcal{W}_{\theta_1, \theta_2}$  form an analytic family of bounded operators w.r.t.  $(\theta_1, \theta_2)$ , while, for fixed values of the parameters,  $\mathcal{W}_{\theta_1, \theta_2}$  induces a similarity between  $Q_{\theta_1, \theta_2}(\mathcal{V})$  and  $Q_{0,0}(\mathcal{V})$ .

**Proposition 3.1** *Let  $\mathcal{V}$  fulfills the conditions (1.9), (2.30) and assume  $(\theta_1, \theta_2) \in \mathcal{B}_\delta((0, 0))$  with  $\delta > 0$  small enough. The operators  $\mathcal{W}_{\theta_1, \theta_2}$  form an analytic family of type A in  $L^2(\mathbb{R})$  w.r.t.  $(\theta_1, \theta_2)$  s.t. the expansion*

$$\mathcal{W}_{\theta_1, \theta_2} = 1 + \mathcal{O}(\theta_1) + \mathcal{O}(\theta_2) \quad (3.3)$$

*holds in the  $\mathcal{L}(L^2(\mathbb{R}), L^2(\mathbb{R}))$  operator norm. The couple  $Q_{\theta_1, \theta_2}(\mathcal{V})$ ,  $Q_{0,0}(\mathcal{V})$  is intertwined through  $\mathcal{W}_{\theta_1, \theta_2}$  by*

$$Q_{\theta_1, \theta_2}(\mathcal{V}) \mathcal{W}_{\theta_1, \theta_2} = \mathcal{W}_{\theta_1, \theta_2} Q_{0,0}(\mathcal{V}). \quad (3.4)$$

**Proof.** For  $\varphi \in L^2(\mathbb{R})$ , let consider the action of  $\mathcal{W}_{\theta_1, \theta_2}$  over  $\varphi$ ; making use of (3.1) and (3.2), this writes as

$$\mathcal{W}_{\theta_1, \theta_2} \varphi = \int_{\mathbb{R}} \frac{dk}{(2\pi)^{1/2}} \psi_-(\cdot, k, \theta_1, \theta_2) (\mathcal{F}_\mathcal{V} \varphi)(k), \quad (3.5)$$

and, expressing  $\psi_-(x, k, \theta_1, \theta_2)$  through the expansion (2.77), we get

$$\begin{aligned} \mathcal{W}_{\theta_1, \theta_2} \varphi &= \int_{\mathbb{R}} \frac{dk}{(2\pi)^{1/2}} \psi_-(\cdot, k) (\mathcal{F}_\mathcal{V} \varphi)(k) + \int_{\mathbb{R}} \frac{dk}{(2\pi)^{1/2}} \left[ \mathcal{O}(\theta_2 k) G^{|k|}(\cdot, b) + \mathcal{O}(\theta_2 k) G^{|k|}(\cdot, a) \right] (\mathcal{F}_\mathcal{V} \varphi)(k) \\ &\quad + \int_{\mathbb{R}} \frac{dk}{(2\pi)^{1/2}} \left[ \mathcal{O}\left(\frac{\theta_1 k}{1+|k|}\right) H^{|k|}(\cdot, b) + \mathcal{O}\left(\frac{\theta_1 k}{1+|k|}\right) H^{|k|}(\cdot, a) \right] (\mathcal{F}_\mathcal{V} \varphi)(k), \end{aligned} \quad (3.6)$$

where, it is important to remark, the symbols denote functions depending only from  $k, \theta_1$  and  $\theta_2$ , but independent from  $x$ . Since  $\int \frac{dk}{(2\pi)^{1/2}} \psi_-(\cdot, k) (\mathcal{F}_\mathcal{V} \varphi)(k) = \mathcal{F}_\mathcal{V}^{-1}(\mathcal{F}_\mathcal{V} \varphi)$ , this equation yields:  $(\mathcal{W}_{\theta_1, \theta_2} - \mathbb{I}) \varphi = I + II$ , where

$$I(\varphi) = \int_{\mathbb{R}} \frac{dk}{(2\pi)^{1/2}} \left[ \mathcal{O}(\theta_2 k) G^{|k|}(\cdot, b) + \mathcal{O}(\theta_2 k) G^{|k|}(\cdot, a) \right] (\mathcal{F}_\mathcal{V} \varphi)(k), \quad (3.7)$$

and

$$II(\varphi) = \int_{\mathbb{R}} \frac{dk}{(2\pi)^{1/2}} \left[ \mathcal{O}\left(\frac{\theta_1 k}{1+|k|}\right) H^{|k|}(\cdot, b) + \mathcal{O}\left(\frac{\theta_1 k}{1+|k|}\right) H^{|k|}(\cdot, a) \right] (\mathcal{F}_\mathcal{V} \varphi)(k). \quad (3.8)$$

In order to obtain the expansion (3.3),  $L^2$ -norm estimates of the maps defined in (3.7) and (3.8) are needed. We consider at first the case of  $I(\varphi)$ ; let  $\varphi \in L^2(\mathbb{R})$  and define  $\phi_\alpha$  as

$$\phi_\alpha(x) = \int_{\mathbb{R}} \frac{dk}{(2\pi)^{1/2}} \mathcal{O}(k) G^{|k|}(x, \alpha) (\mathcal{F}_V \varphi)(k), \quad \alpha \in \{a, b\}, \quad (3.9)$$

being  $\mathcal{O}(\cdot)$  depending only on  $k$ . The  $L^2$ -norm of  $\phi_\alpha$  is bounded by

$$\|\phi_\alpha\|_{L^2(\mathbb{R})} \leq \|1_{\{x \leq a\}} \phi_\alpha\|_{L^2(\mathbb{R})} + \|1_{(a,b)} \phi_\alpha\|_{L^2(\mathbb{R})} + \|1_{\{x \geq b\}} \phi_\alpha\|_{L^2(\mathbb{R})}. \quad (3.10)$$

We consider the case:  $\alpha = b$ ; making use of the explicit form of  $G^k(x, b)$ , given by (2.25) for  $\zeta = k$ , and exploiting the relations (2.19) and

$$1_{\{x \leq a\}} \chi_-(x, k) = e^{-ikx}, \quad 1_{\{x \geq b\}} \chi_+(x, k) = e^{-ikx}, \quad (3.11)$$

we have

$$1_{\{x \leq a\}}(x) \mathcal{O}(k) G^{|k|}(x, b) = 1_{\{x \leq a\}}(x) \tau_1(k) e^{-i|k|x} \quad (3.12)$$

$$1_{\{x \geq b\}}(x) \mathcal{O}(k) G^{|k|}(x, b) = 1_{\{x \geq b\}}(x) \tau_2(k) e^{i|k|x} \quad (3.13)$$

with  $\tau_1, \tau_2 \in L_k^\infty(\mathbb{R})$ . Let  $\mathcal{P}$  be the 'parity operator', defined according to:  $\mathcal{P}u(t) = u(-t)$ ; from (3.12), it follows

$$\begin{aligned} 1_{\{x \leq a\}}(x) \phi_b(x) &= 1_{\{x \leq a\}}(x) \int_{\mathbb{R}} \frac{dk}{(2\pi)^{1/2}} \tau_1(k) e^{-i|k|x} (\mathcal{F}_V \varphi)(k) \\ &= 1_{\{x \leq a\}}(x) (\mathcal{F}_0^{-1} (1_{k < 0} \tau_1 \mathcal{F}_V \varphi + \mathcal{P} (1_{k > 0} \tau_1 \mathcal{F}_V \varphi))) (x), \end{aligned} \quad (3.14)$$

where, according to the notation introduced in (3.1),  $\mathcal{F}_0$  is the standard Fourier transform. Thus,  $1_{\{x \leq a\}} \phi_b$  is estimated by

$$\|1_{\{x \leq a\}} \phi_b\|_{L^2(\mathbb{R})} = \|\mathcal{F}_0^{-1} (1_{k < 0} \tau_1 \mathcal{F}_V \varphi)\|_{L^2(\mathbb{R})} + \|\mathcal{F}_0^{-1} \mathcal{P} (1_{k > 0} \tau_1 \mathcal{F}_V \varphi)\|_{L^2(\mathbb{R})} \lesssim \|\varphi\|_{L^2(\mathbb{R})}, \quad (3.15)$$

while, for  $1_{\{x \geq b\}} \phi_b$ , a similar inequality follows by using (3.13)

$$\|1_{\{x \geq b\}} \phi_b\|_{L^2(\mathbb{R})} = \|\mathcal{F}_0^{-1} \mathcal{P} (1_{k < 0} \tau_2 \mathcal{F}_V \varphi)\|_{L^2(\mathbb{R})} + \|\mathcal{F}_0^{-1} (1_{k > 0} \tau_2 \mathcal{F}_V \varphi)\|_{L^2(\mathbb{R})} \lesssim \|\varphi\|_{L^2(\mathbb{R})}. \quad (3.16)$$

According to definition of  $G^k(x, b)$  for  $x < b$ , the term  $1_{(a,b)} \phi_b$  writes as

$$1_{(a,b)}(x) \phi_b(x) = 1_{(a,b)}(x) \int_{\mathbb{R}} dk \frac{\mathcal{O}(k) \chi_+(b, |k|)}{w(|k|)} \chi_-(x, |k|) (\mathcal{F}_V \varphi)(k) = 1_{(a,b)}(x) \int_{\mathbb{R}} dk \chi_-(x, |k|) \tau_3(k) (\mathcal{F}_V \varphi)(k), \quad (3.17)$$

where  $\tau_3 \in L_k^\infty(\mathbb{R})$  is:  $\tau_3(k) = \frac{\mathcal{O}(k) \chi_+(b, |k|)}{w(|k|)}$ . Using the definition (2.60) and the identities

$$\chi_\pm(\cdot, -k) = \chi_\pm^*(\cdot, k), \quad w(-k) = w^*(k), \quad (3.18)$$

it follows

$$1_{k < 0}(k) \chi_-(x, -k) = 1_{k < 0}(k) \frac{w(-k)}{2ik} \psi_-(x, k), \quad (3.19)$$

$$1_{k \geq 0}(k) \chi_-(x, k) = -1_{k \geq 0}(k) \frac{w(k)}{2ik} \psi_-(x, -k). \quad (3.20)$$

Take  $\tilde{\tau}_3(k) = \tau_3(k) \frac{w(-k)}{2ik}$  and  $\hat{\tau}_3(k) = \tau_3(k) \frac{w(k)}{-2ik}$ ; it results:  $\tilde{\tau}_3, \hat{\tau}_3 \in L_k^\infty(\mathbb{R})$  and the r.h.s. of (3.17) rephrases as

$$1_{(a,b)}(x) \phi_b(x) = 1_{(a,b)}(x) \left[ \int_{k < 0} dk \psi_-(x, k) \tilde{\tau}_3(k) (\mathcal{F}_V \varphi)(k) + \int_{k > 0} dk \psi_-(x, -k) (\hat{\tau}_3(k) (\mathcal{F}_V \varphi)(k)) \right]. \quad (3.21)$$

The first term identifies with the inverse Fourier transform of  $1_{k < 0} \tilde{\tau}_3 \mathcal{F}_V \varphi$ ,

$$\int_{k < 0} dk \psi_-(\cdot, k) \tilde{\tau}_3(k) (\mathcal{F}_V \varphi)(k) = \mathcal{F}_V^{-1} (1_{k < 0} \tilde{\tau}_3 \mathcal{F}_V \varphi), \quad (3.22)$$



while, for the second term, we have

$$\int_{k>0} dk \psi_-(\cdot, -k) (\hat{\tau}_3(k) (\mathcal{F}_V \varphi)(k)) = - \int_{k<0} dk \psi_-(\cdot, k) (\mathcal{P}(\hat{\tau}_3 \mathcal{F}_V \varphi))(k) = -\mathcal{F}_V^{-1} \mathcal{P}(1_{k>0} \hat{\tau}_3 \mathcal{F}_V \varphi). \quad (3.23)$$

The above relations yield the estimate

$$\|1_{(a,b)} \phi_b\|_{L^2(\mathbb{R})} = \|1_{(a,b)} \mathcal{F}_V^{-1} (1_{k<0} \tilde{\tau}_3 \mathcal{F}_V \varphi)^*\|_{L^2(\mathbb{R})} + \|1_{(a,b)} \mathcal{F}_V^{-1} \mathcal{P}(1_{k>0} \tilde{\tau}_3 \mathcal{F}_V \varphi)\|_{L^2(\mathbb{R})} \lesssim \|\varphi\|_{L^2(\mathbb{R})}. \quad (3.24)$$

As a consequence of (3.15), (3.16) and (3.24) we get

$$\|\phi_b\|_{L^2(\mathbb{R})} \lesssim \|\varphi\|_{L^2(\mathbb{R})}, \quad (3.25)$$

and a similar computation in the case of  $\phi_a$  leads us to:  $\|\phi_a\|_{L^2(\mathbb{R})} \lesssim \|\varphi\|_{L^2(\mathbb{R})}$ . According to the definitions (3.7) and (3.9), it follows

$$\|I(\varphi)\|_{L^2(\mathbb{R})} \lesssim |\theta_2| \left( \|\phi_a\|_{L^2(\mathbb{R})} + \|\phi_b\|_{L^2(\mathbb{R})} \right) \lesssim |\theta_2| \|\varphi\|_{L^2(\mathbb{R})}. \quad (3.26)$$

For the map  $II(\varphi)$ , we introduce  $\psi_\alpha$  defined as

$$\psi_\alpha(x) = \int_{\mathbb{R}} dk \mathcal{O}\left(\frac{k}{1+|k|}\right) H^{|k|}(x, \alpha) (\mathcal{F}_V \varphi)(k), \quad \alpha \in \{a, b\}, \quad (3.27)$$

where  $\mathcal{O}(\cdot)$  depends only on  $k$ . For  $\alpha = b$ , the explicit form of  $H^k(x, b)$ , given by (2.26) for  $\zeta = k$ , and the relations (2.19), (3.11), yield

$$1_{\{x \leq a\}}(x) \mathcal{O}\left(\frac{k}{1+|k|}\right) H^{|k|}(x, b) = 1_{\{x \leq a\}}(x) \eta_1(k) e^{-i|k|x} \quad (3.28)$$

$$1_{(a,b)}(x) \mathcal{O}\left(\frac{k}{1+|k|}\right) H^{|k|}(x, b) = 1_{(a,b)}(x) \eta_3(k) \chi_-(x, |k|) \quad (3.29)$$

$$1_{\{x \geq b\}}(x) \mathcal{O}\left(\frac{k}{1+|k|}\right) H^{|k|}(x, b) = 1_{\{x \geq b\}}(x) \eta_2(k) e^{i|k|x} \quad (3.30)$$

where  $\eta_{i=1,2,3} \in L_k^\infty(\mathbb{R})$  are described by  $\mathcal{O}\left(\frac{k}{1+|k|}\right)$ . Setting:  $\tilde{\eta}_3(k) = \eta_3(k) \frac{w(-k)}{2ik}$  and  $\hat{\eta}_3(k) = \eta_3(k) \frac{w(k)}{-2ik}$  (which, according to the characterization of  $\eta_3$ , still implies:  $\tilde{\eta}_3, \hat{\eta}_3 \in L_k^\infty(\mathbb{R})$ ), and proceeding as before, we obtain the decomposition

$$\begin{aligned} \psi_b &= 1_{\{x \leq a\}} [\mathcal{F}_0^{-1} (1_{k<0} \eta_1 \mathcal{F}_V \varphi + \mathcal{P}(1_{k>0} \eta_1 \mathcal{F}_V \varphi))] \\ &\quad + 1_{\{x \geq b\}} [\mathcal{F}_0^{-1} \mathcal{P}(1_{k<0} \eta_2 \mathcal{F}_V \varphi) + \mathcal{F}_0^{-1} (1_{k>0} \eta_2 \mathcal{F}_V \varphi)] \\ &\quad + 1_{(a,b)} [\mathcal{F}_V^{-1} (1_{k<0} \tilde{\eta}_3 \mathcal{F}_V \varphi) - \mathcal{F}_V^{-1} \mathcal{P}(1_{k>0} \hat{\eta}_3 \mathcal{F}_V \varphi)]. \end{aligned} \quad (3.31)$$

This entails:  $\|\psi_b\|_{L^2(\mathbb{R})} \lesssim \|\varphi\|_{L^2(\mathbb{R})}$ , while, with similar computations, the corresponding estimate in the case of  $\psi_a$  is obtained. From the definitions (3.8) and (3.27), follows

$$\|II\|_{L^2(\mathbb{R})} \lesssim |\theta_1| \left( \|\psi_a\|_{L^2(\mathbb{R})} + \|\psi_b\|_{L^2(\mathbb{R})} \right) \lesssim |\theta_1| \|\varphi\|_{L^2(\mathbb{R})}. \quad (3.32)$$

The expansion (3.3) is a consequence of (3.26) and (3.32). Since the symbols in (3.7)-(3.8) are holomorphic in  $(\theta_1, \theta_2)$ ,  $\mathcal{W}_{\theta_1, \theta_2}$  forms an analytic family of type A w.r.t. the parameters.

Next, we consider the relation (3.4). Let  $\varphi \in D(Q_{0,0}(\mathcal{V}))$ , using the functional calculus of  $Q_{0,0}(\mathcal{V})$ , we have:  $(\mathcal{F}_V(Q_{0,0}(\mathcal{V})\varphi))(k) = k^2 (\mathcal{F}_V \varphi)(k)$  and, according to (3.5), the r.h.s. of (3.4) writes as

$$\mathcal{W}_{\theta_1, \theta_2} Q_{0,0}(\mathcal{V})\varphi = \int_{\mathbb{R}} dk \psi_-(\cdot, k, \theta_1, \theta_2) k^2 (\mathcal{F}_V \varphi)(k). \quad (3.33)$$

To discuss the action of  $Q_{\theta_1, \theta_2}(\mathcal{V})\mathcal{W}_{\theta_1, \theta_2}$  over  $D(Q_{0,0}(\mathcal{V}))$ , we use the expansion

$$\mathcal{W}_{\theta_1, \theta_2} \varphi = \varphi + I(\varphi) + II(\varphi). \quad (3.34)$$

From the above results, the map  $I(\varphi) + II(\varphi)$  can be represented as

$$\begin{aligned} I(\varphi) + II(\varphi) = & 1_{\{x \leq a\}} [\mathcal{F}_0^{-1} (1_{k < 0} \mu_1 \mathcal{F}_V \varphi + \mathcal{P} (1_{k > 0} \mu_1 \mathcal{F}_V \varphi))] \\ & + 1_{\{x \geq b\}} [\mathcal{F}_0^{-1} \mathcal{P} (1_{k < 0} \mu_2 \mathcal{F}_V \varphi) + \mathcal{F}_0^{-1} (1_{k > 0} \mu_2 \mathcal{F}_V \varphi)] \\ & + 1_{(a,b)} [\mathcal{F}_V^{-1} (1_{k < 0} \mu_3 \mathcal{F}_V \varphi) - \mathcal{F}_V^{-1} (1_{k < 0} (\mathcal{P} (\mu_4 \mathcal{F}_V \varphi)))] , \end{aligned} \quad (3.35)$$

where  $\mu_i \in L_k^\infty(\mathbb{R})$ ,  $i = 1, \dots, 4$ , are suitable bounded functions of  $k$ . Let  $u \in L_k^\infty(\mathbb{R})$  and  $\mathcal{V}, \mathcal{V}'$  any couple of potentials fulfilling the assumptions; the operators  $\mathcal{F}_{\mathcal{V}'}^{-1} u \mathcal{F}_V$  and  $\mathcal{F}_{\mathcal{V}'}^{-1} \mathcal{P} u \mathcal{F}_V$  map  $D(Q_{0,0}(\mathcal{V}))$  into itself. Then, as a consequence of (3.34), (3.35), the operator  $\mathcal{W}_{\theta_1, \theta_2}$  maps  $D(Q_{0,0}(\mathcal{V}))$  into  $D(Q(\mathcal{V}))$ , while, according to (2.76) and (3.33),  $\mathcal{W}_{\theta_1, \theta_2} \varphi$  fulfills the interface conditions (1.10) for all  $\varphi \in D(Q_{0,0}(\mathcal{V}))$ ; we obtain:  $\mathcal{W}_{\theta_1, \theta_2} \in \mathcal{L}(D(Q_{0,0}(\mathcal{V})), D(Q_{\theta_1, \theta_2}(\mathcal{V})))$ . Moreover, from the relation:  $(Q_{\theta_1, \theta_2}(\mathcal{V}) - k^2) \psi_-(\cdot, k, \theta_1, \theta_2) = 0$ , it follows

$$Q_{\theta_1, \theta_2}(\mathcal{V}) \mathcal{W}_{\theta_1, \theta_2} \varphi = \int_{\mathbb{R}} dk \psi_-(\cdot, k, \theta_1, \theta_2) k^2 (\mathcal{F}_V \varphi)(k), \quad (3.36)$$

which lead us to (3.4). ■

### 3.1 Proof of the Theorem 1.2.

When the parameters  $\theta_1, \theta_2$  are chosen in a suitably close neighbourhood of the origin, the expansion (3.3) yields

$$\mathcal{W}_{\theta_1, \theta_2}^{-1} = 1 + \mathcal{O}(\theta_1) + \mathcal{O}(\theta_2), \quad (3.37)$$

and  $Q_{\theta_1, \theta_2}(\mathcal{V})$  expresses as the conjugated operator

$$Q_{\theta_1, \theta_2}(\mathcal{V}) = \mathcal{W}_{\theta_1, \theta_2} Q_{0,0}(\mathcal{V}) \mathcal{W}_{\theta_1, \theta_2}^{-1}. \quad (3.38)$$

Let us introduce

$$U_{\theta_1, \theta_2}(t) = \mathcal{W}_{\theta_1, \theta_2} U_{0,0}(t) \mathcal{W}_{\theta_1, \theta_2}^{-1}, \quad (3.39)$$

being  $U_{0,0}(t) = e^{-itQ_{0,0}(\mathcal{V})}$  the unitary propagator associated to  $-iQ_{0,0}(\mathcal{V})$ . Due to the properties of  $\mathcal{W}_{\theta_1, \theta_2}$ ,  $U_{\theta_1, \theta_2}(t)$  is holomorphic w.r.t.  $(\theta_1, \theta_2)$ , while, for fixed values of the parameters, the family  $\{U_{\theta_1, \theta_2}(t)\}_{t \in \mathbb{R}}$  forms a strongly continuous group on  $L^2(\mathbb{R})$  and, according to (3.38), (3.39), we have

$$i\partial_t (U_{\theta_1, \theta_2}(t)u) = Q_{\theta_1, \theta_2}(\mathcal{V}) U_{\theta_1, \theta_2}(t)u, \quad (3.40)$$

for all  $u \in L^2(\mathbb{R})$ . This allows to identify  $U_{\theta_1, \theta_2}(t)$  with the quantum dynamical system generated by  $-iQ_{\theta_1, \theta_2}(\mathcal{V})$ . Making use of (3.3) and (3.37), we get

$$U_{\theta_1, \theta_2}(t) = U_{0,0}(t) + \mathcal{R}(t, \theta_1, \theta_2), \quad (3.41)$$

where the remainder term is strongly continuous and uniformly bounded w.r.t.  $t$  in the  $L^2$ -operator norm, allowing the representation:  $\mathcal{R}(t, \theta_1, \theta_2) = \mathcal{O}(\theta_1) + \mathcal{O}(\theta_2)$ .

### 3.2 Time dependent wave operators and scattering systems.

So far, we have investigated the continuity of the dynamical system generated by  $-iQ_{\theta_1, \theta_2}(\mathcal{V})$  w.r.t. the parameters  $\theta_{i=1,2}$ . This has been analyzed by using small- $\theta_i$  expansions of the 'stationary waves operators'  $\mathcal{W}_{\theta_1, \theta_2}$  defined in (3.2). In what follows we consider the scattering problem for the pair  $\{Q_{\theta_1, \theta_2}(\mathcal{V}), Q_{0,0}(\mathcal{V})\}$  and show that  $\mathcal{W}_{\theta_1, \theta_2}$  coincides with a wave operator of this couple. The next Lemma discusses this point under the assumptions of the Proposition 3.1.

**Lemma 3.2** *Let  $\mathcal{V}$  fulfills the conditions (1.9), (2.30) and assume  $(\theta_1, \theta_2) \in \mathcal{B}_\delta((0,0))$  with  $\delta > 0$  small enough. Then*

$$\text{s-} \lim_{t \rightarrow -\infty} e^{itQ_{\theta_1, \theta_2}(\mathcal{V})} e^{-itQ_{0,0}(\mathcal{V})} = \mathcal{W}_{\theta_1, \theta_2}. \quad (3.42)$$

**Proof.** Let introduce the modified transform  $\mathcal{F}_{\mathcal{V}, \theta_1, \theta_2}$  defined according to

$$\mathcal{F}_{\mathcal{V}, \theta_1, \theta_2}^{-1} f = \int_{\mathbb{R}} dk \psi_-(x, k, \theta_1, \theta_2) f(k), \quad f \in L^2(\mathbb{R}). \quad (3.43)$$

The action of  $\mathcal{W}_{\theta_1, \theta_2}$  can be expressed in terms of  $\mathcal{F}_{\mathcal{V}, \theta_1, \theta_2}$  and  $\mathcal{F}_V$  as:  $\mathcal{W}_{\theta_1, \theta_2} = \mathcal{F}_{\mathcal{V}, \theta_1, \theta_2}^{-1} \mathcal{F}_V$ , from which we get:  $\mathcal{F}_{\mathcal{V}, \theta_1, \theta_2} = \mathcal{F}_V \mathcal{W}_{\theta_1, \theta_2}^{-1}$ . According to the expansion (3.3), it results

$$\mathcal{F}_{\mathcal{V}, \theta_1, \theta_2} = \mathcal{F}_V (1 + \mathcal{O}(\theta_1) + \mathcal{O}(\theta_2)) \quad (3.44)$$

in the  $L^2$ -operator norm sense. Making use of the intertwining property, it follows:  $\mathcal{W}_{\theta_1, \theta_2}^*(Q_{\theta_1, \theta_2}(\mathcal{V}))^* = Q_{0,0}(\mathcal{V})\mathcal{W}_{\theta_1, \theta_2}^*$ . Since  $\mathcal{W}_{\theta_1, \theta_2}^* = \mathcal{F}_{\mathcal{V}}^{-1} \left( \mathcal{F}_{\mathcal{V}, \theta_1, \theta_2}^{-1} \right)^*$  and  $(Q_{\theta_1, \theta_2}(\mathcal{V}))^* = Q_{-\theta_2^*, -\theta_1^*}(\mathcal{V})$  (see eq. (1.12)), we have

$$\mathcal{F}_{\mathcal{V}}^{-1} \left( \mathcal{F}_{\mathcal{V}, \theta_1, \theta_2}^{-1} \right)^* Q_{-\theta_2^*, -\theta_1^*}(\mathcal{V}) = Q_{0,0}(\mathcal{V})\mathcal{F}_{\mathcal{V}}^{-1} \left( \mathcal{F}_{\mathcal{V}, \theta_1, \theta_2}^{-1} \right)^*. \quad (3.45)$$

Let us denote with  $A$  the operator of multiplication by  $k^2$ . Using the functional calculus for  $Q_{0,0}(\mathcal{V})$ , this operator is represented by:  $A = \mathcal{F}_{\mathcal{V}} Q_{0,0}(\mathcal{V})\mathcal{F}_{\mathcal{V}}^{-1}$ , and the previous relation rephrases as:  $\left( \mathcal{F}_{\mathcal{V}, \theta_1, \theta_2}^{-1} \right)^* Q_{-\theta_2^*, -\theta_1^*}(\mathcal{V}) = A \left( \mathcal{F}_{\mathcal{V}, \theta_1, \theta_2}^{-1} \right)^*$ . Then, taking the adjoint, yields

$$Q_{\theta_1, \theta_2}(\mathcal{V})\mathcal{F}_{\mathcal{V}, \theta_1, \theta_2}^{-1} = \mathcal{F}_{\mathcal{V}, \theta_1, \theta_2}^{-1} A. \quad (3.46)$$

To identify  $\mathcal{W}_{\theta_1, \theta_2}$  with a wave operator, according to the time dependent definition

$$W_{-}(Q_{\theta_1, \theta_2}(\mathcal{V}), Q_{0,0}(\mathcal{V})) = \text{s-} \lim_{t \rightarrow -\infty} e^{itQ_{\theta_1, \theta_2}(\mathcal{V})} e^{-itQ_{0,0}(\mathcal{V})}, \quad (3.47)$$

it is enough to prove that

$$\lim_{t \rightarrow -\infty} \left\| \left( e^{itQ_{\theta_1, \theta_2}(\mathcal{V})} e^{-itQ_{0,0}(\mathcal{V})} - \mathcal{W}_{\theta_1, \theta_2} \right) u \right\|_{L^2(\mathbb{R})} = 0. \quad (3.48)$$

Explicitly, the function in (3.48) reads as

$$\left( e^{itQ_{\theta_1, \theta_2}(\mathcal{V})} e^{-itQ_{0,0}(\mathcal{V})} - \mathcal{F}_{\mathcal{V}, \theta_1, \theta_2}^{-1} \mathcal{F}_{\mathcal{V}} \right) u. \quad (3.49)$$

Setting  $g = \mathcal{F}_{\mathcal{V}} u$ , we have:  $e^{-itQ_{0,0}(\mathcal{V})} u = \mathcal{F}_{\mathcal{V}}^{-1} e^{-itA} g$ , and (3.49) rephrases as

$$e^{itQ_{\theta_1, \theta_2}(\mathcal{V})} \left( \mathcal{F}_{\mathcal{V}}^{-1} e^{-itA} - e^{-itQ_{\theta_1, \theta_2}(\mathcal{V})} \mathcal{F}_{\mathcal{V}, \theta_1, \theta_2}^{-1} \right) g. \quad (3.50)$$

Then, using (3.46) and the definitions (3.1), (3.43), we get

$$\begin{aligned} \left( e^{itQ_{\theta_1, \theta_2}(\mathcal{V})} e^{-itQ_{0,0}(\mathcal{V})} - \mathcal{W}_{\theta_1, \theta_2} \right) u &= e^{itQ_{\theta_1, \theta_2}(\mathcal{V})} \left( \mathcal{F}_{\mathcal{V}}^{-1} - \mathcal{F}_{\mathcal{V}, \theta_1, \theta_2}^{-1} \right) e^{-itA} g \\ &= e^{itQ_{\theta_1, \theta_2}(\mathcal{V})} \int_{\mathbb{R}} dk \left( \psi_{-}(\cdot, k) - \psi_{-}(\cdot, k, \theta_1, \theta_2) \right) e^{-itk^2} g(k), \quad g \in \mathcal{F}_{\mathcal{V}} u. \end{aligned} \quad (3.51)$$

Under our assumptions, the result of the Corollary 2.6 applies and the r.h.s. of (3.51) can be further developed through the expansion (2.77). This yields

$$\begin{aligned} \left( e^{itQ_{\theta_1, \theta_2}(\mathcal{V})} e^{-itQ_{0,0}(\mathcal{V})} - \mathcal{W}_{\theta_1, \theta_2} \right) u &= \\ &= \int_{\mathbb{R}} dk \mathcal{O}(\theta_2 k) G^{\sigma k}(\cdot, b) e^{-itk^2} g(k) + \int_{\mathbb{R}} dk \mathcal{O} \left( \frac{\theta_1 k}{1 + |k|} \right) H^{\sigma k}(\cdot, b) e^{-itk^2} g(k) \\ &+ \int_{\mathbb{R}} dk \mathcal{O}(\theta_2 k) G^{\sigma k}(\cdot, a) e^{-itk^2} g(k) + \int_{\mathbb{R}} dk \mathcal{O} \left( \frac{\theta_1 k}{1 + |k|} \right) H^{\sigma k}(\cdot, a) e^{-itk^2} g(k). \end{aligned} \quad (3.52)$$

where  $\sigma = \frac{k}{|k|}$ , while the functions  $\mathcal{O}(\theta_2 k)$  and  $\mathcal{O} \left( \frac{\theta_1 k}{1 + |k|} \right)$  are independent from  $x$ . To obtain (3.42), it is enough to show that, for all  $g \in C_0^\infty(\mathbb{R})$ , limits of the type (3.48) are available for each term at the r.h.s. of (3.52). In what follows, we consider the first contribution of (3.52), while the other terms can be treated within the same approach. Since we work with fixed  $(\theta_1, \theta_2)$ , the dependence from these parameters is next omitted. Our aim is to prove

$$\lim_{|t| \rightarrow \infty} \left\| \int_{\mathbb{R}} dk \mathcal{O}(k) G^{\sigma k}(\cdot, b) e^{-itk^2} g(k) \right\|_{L^2(\mathbb{R})} = 0,$$

when  $g \in C_0^\infty(\mathbb{R})$ . We use the definitions (2.25), (2.31), with  $\zeta = k$ , to write

$$\left\| \int_{\mathbb{R}} dk \mathcal{O}(k) G^{\sigma k}(\cdot, b) e^{-itk^2} g(k) \right\|_{L^2(\mathbb{R})} = I + II \quad (3.53)$$

where

$$I = \int_{-\infty}^b dx \left| \int_{\mathbb{R}} dk \frac{\mathcal{O}(k)}{w(|k|)} \chi_{-}(x, |k|) \chi_{+}(b, |k|) e^{-itk^2} g(k) \right|^2, \quad (3.54)$$

$$II = \int_b^{+\infty} dx \left| \int_{\mathbb{R}} dk \frac{\mathcal{O}(k)}{w(|k|)} \chi_{+}(x, |k|) \chi_{-}(b, |k|) e^{-itk^2} g(k) \right|^2. \quad (3.55)$$

Recall that  $\chi_{\pm}$  are defined through the equations

$$1_{x < b}(x)\chi_+(x, k) = e^{ikx} + \int_x^b \frac{\sin k(t-x)}{k} \mathcal{V}(t)\chi_+(t, k) dt, \quad 1_{x \geq b}(x)\chi_+(x, k) = e^{ikx}, \quad (3.56)$$

$$1_{x > a}(x)\chi_-(x, k) = e^{-ikx} - \int_a^x \frac{\sin k(t-x)}{k} \mathcal{V}(t)\chi_-(t, k) dt, \quad 1_{x \leq a}(x)\chi_-(x, k) = e^{-ikx}, \quad (3.57)$$

and introduce the functions

$$\gamma_+(x) = \int_x^b |\mathcal{V}(t)| dt, \quad \gamma_-(x) = \int_a^x |\mathcal{V}(t)| dt. \quad (3.58)$$

For compactly supported short-range potentials (see (1.9)), it results:  $\gamma_+ \in L^2((a, +\infty))$  and  $\gamma_- \in L^2((-\infty, b))$ . As stated in the Proposition 2.1,  $\chi_{\pm}$  are uniformly bounded w.r.t.  $x, k \in \mathbb{R}$  and the previous relations rephrase as

$$1_{x < b}(x)\chi_+(x, k) = e^{ikx} + \ell_+(x, k), \quad 1_{x \geq b}(x)\chi_+(x, k) = e^{ikx}, \quad (3.59)$$

$$1_{x > a}(x)\chi_-(x, k) = e^{-ikx} + \ell_-(x, k), \quad 1_{x \leq a}(x)\chi_-(x, k) = e^{-ikx}, \quad (3.60)$$

with  $\ell_{\pm}$  s.t.:

$$|\ell_{\pm}(x, k)| < \frac{1}{k} \|\chi_{\pm}\|_{L_{x,k}^{\infty}(\mathbb{R}^2)} \gamma_{\pm}(x). \quad (3.61)$$

Plugging these relations into (3.54)-(3.55) and using  $1_{x > b}(x)g_+(x, k) = 0$ , we get

$$I \leq \int_{-\infty}^b dx \left| \int_{\mathbb{R}} dk e^{-i|k|x} e^{-itk^2} \frac{\mathcal{O}(k) e^{i|k|b}}{w(|k|)} g(k) \right|^2 + \int_{-\infty}^b dx \left| \int_{\mathbb{R}} dk \ell_-(x, |k|) e^{-itk^2} \frac{\mathcal{O}(k) e^{i|k|b}}{w(|k|)} g(k) \right|^2, \quad (3.62)$$

$$II = \int_b^{+\infty} dx \left| \int_{\mathbb{R}} dk e^{i|k|x} e^{-itk^2} \frac{\mathcal{O}(k)}{w(|k|)} \chi_-(b, |k|) g(k) \right|^2. \quad (3.63)$$

With the change of variable:  $s = -x + b$ , the first integral at the r.h.s. of (3.62) writes as

$$\int_{-\infty}^b dx \left| \int_{\mathbb{R}} dk e^{-i|k|x} e^{-itk^2} \frac{\mathcal{O}(k) e^{i|k|b}}{w(|k|)} g(k) \right|^2 = \int_0^{+\infty} ds \left| \int_{\mathbb{R}} dk e^{i|k|s} e^{-itk^2} \left( \frac{\mathcal{O}(k)}{w(|k|)} g(k) \right) \right|^2, \quad (3.64)$$

while, setting:  $s = x - b$ , the equation (3.63) rephrases as

$$II = \int_0^{+\infty} ds \left| \int_{\mathbb{R}} dk e^{i|k|s} e^{-itk^2} \left( \frac{\mathcal{O}(k) e^{i|k|b}}{w(|k|)} \chi_-(b, |k|) g(k) \right) \right|^2. \quad (3.65)$$

Due to the relation  $w(k) = \mathcal{O}(1+k)$  (see the proof of Lemma 2.2), the functions  $\frac{\mathcal{O}(k)}{w(k)} g(k)$  and  $\frac{\mathcal{O}(k) e^{i|k|b}}{w(|k|)} \chi_-(b, |k|) g(k)$  appearing above both belong to  $L_k^2(\mathbb{R})$ . Moreover, as a consequence of (3.61), it results

$$\left| \ell_-(x, |k|) \frac{\mathcal{O}(k) e^{i|k|b}}{w(|k|)} \right| \leq \|\chi_-\|_{L_{x,k}^{\infty}(\mathbb{R}^2)} \left| \frac{\mathcal{O}(k)}{kw(|k|)} \right| \gamma_-(x) \lesssim \gamma_-(x) \in L^2((-\infty, b)), \quad (3.66)$$

We get

$$I \leq \int_0^{+\infty} ds \left| \int_{\mathbb{R}} dk e^{i|k|s} e^{-itk^2} q_1(k) \right|^2 + \int_{-\infty}^b dx \left| \int_{\mathbb{R}} dk e^{-itk^2} q_2(k, x) g(k) \right|^2, \quad (3.67)$$

$$II = \int_0^{+\infty} ds \left| \int_{\mathbb{R}} dk e^{i|k|s} e^{-itk^2} q_3(k) \right|^2. \quad (3.68)$$

where, according to the previous remarks,  $q_1, q_3 \in L_k^2(\mathbb{R})$ , while  $q_2$  allows the estimate

$$|q_2(k, x)| \leq f(x) \in L^2((-\infty, b)) . \quad (3.69)$$

Then, it follows from an application of the Lemma 2.6.4 of [30] that

$$\lim_{t \rightarrow -\infty} \int_0^{+\infty} ds \left| \int_{\mathbb{R}} dk e^{i|k|s - itk^2} q_j(k) \right|^2 = 0, \quad j = 1, 3, \quad (3.70)$$

Moreover, for  $g \in \mathcal{C}_0^\infty(\mathbb{R})$ , the Riemann-Lebesgue Lemma implies

$$\lim_{|t| \rightarrow \infty} \int_{\mathbb{R}} dk e^{-itk^2} q_2(k, x) g(k) = 0; \quad (3.71)$$

thus, using the estimate

$$\left| \int_{\mathbb{R}} dk e^{-itk^2} q_2(k, x) g(k) \right| \lesssim |f(x)| \left( \int_{\mathbb{R}} dk |g(k)| \right) \lesssim |f(x)| ,$$

and the dominated convergence theorem, we get

$$\lim_{|t| \rightarrow \infty} \left\| \int_{\mathbb{R}} dk e^{-itk^2} q_2(k, x) g(k) \right\|_{L^2(-\infty, b)} = 0. \quad (3.72)$$

■

The above result exploits the condition:  $(\theta_1, \theta_2) \in \mathcal{B}_\delta((0, 0))$  which has been previously introduced to identify the spectra of the operators  $Q_{\theta_1, \theta_2}(\mathcal{V})$  and  $Q_{0,0}(\mathcal{V})$ , and to compare the corresponding quantum dynamics. Nevertheless a question is left open: does a small parameter condition is needed in order that the pair  $\{Q_{\theta_1, \theta_2}(\mathcal{V}), Q_{0,0}(\mathcal{V})\}$  forms a complete scattering system? Actually this restriction does not seems to be necessary. It has been shown, indeed, that a key point in the development of the scattering theory for the possibly non-selfadjoint pair  $\{H_0, H_1\}$  is the existence of the strong limit on the real axis of the characteristic functions associated to  $H_{i=0,1}$  (e.g. in [27] and [28]). In particular, the Theorem 4.1 in [27] makes use of this assumption to study the existence of the related waves operators. According to [26], the resolvent formula (2.15) implies that, for any  $(\theta_1, \theta_2)$ , the characteristic function of the operator  $Q_{\theta_1, \theta_2}(\mathcal{V})$  has boundary values a.e. on the real axis (for this point we refer to the last Proposition in [26] and to the references therein). The above remark suggest the possibility of defining the scattering system  $\{Q_{\theta_1, \theta_2}(\mathcal{V}), Q_{0,0}(\mathcal{V})\}$  without restrictions on  $(\theta_1, \theta_2)$ .

A slightly different approach to the scattering problem consists in characterizing the scattering matrix for  $\{Q_{\theta_1, \theta_2}(\mathcal{V}), Q_{0,0}(\mathcal{V})\}$ . In the case of selfadjoint extensions of a symmetric operator, a relation between the scattering matrix and the Weyl function, associated with a boundary triple, have been established in [3]. An extension of this result to the case of non-selfadjoint extensions would represents an useful insight in the study of the scattering properties of the system  $\{Q_{\theta_1, \theta_2}(\mathcal{V}), Q_{0,0}(\mathcal{V})\}$ .

## 4 The regime of quantum wells in a semiclassical island.

This section is concerned with the  $h$ -dependent model introduced in (1.18). Our aim is to extend the previous analysis taking into account the rôle of the scaling parameter  $h$  in the definition of the modified dynamics. A first step consists in the study of the Jost's solutions, the generalized eigenfunctions and the Green's kernel associated to the operators  $Q_{0,0}^h(\mathcal{V})$ , which, according to the definition (1.18), are given by

$$Q_{0,0}^h(\mathcal{V}) : \begin{cases} D(Q_{0,0}^h(\mathcal{V})) = H^2(\mathbb{R}) , \\ (Q_{0,0}^h(\mathcal{V}) u)(x) = -h^2 u''(x) + \mathcal{V}(x) u(x), \quad x \in \mathbb{R}, \end{cases} \quad (4.1)$$

with  $\mathcal{V}$  locally supported on  $(a, b)$ . In what follows,  $\chi_\pm^h(\cdot, \zeta, \mathcal{V})$  denote the solutions of the equation

$$(-h^2 \partial_x^2 + \mathcal{V}) u = \zeta^2 u, \quad (4.2)$$

fulfilling the conditions

$$\chi_+^h(\cdot, \zeta, \mathcal{V})|_{x>b} = e^{i\frac{\zeta}{h}x}, \quad \chi_-^h(\cdot, \zeta, \mathcal{V})|_{x<a} = e^{-i\frac{\zeta}{h}x}. \quad (4.3)$$

**Proposition 4.1** *For fixed  $h > 0$ , assume  $\mathcal{V}$  to be defined according to (1.9). The solutions  $\chi_\pm^h$  of the problem (4.2)-(4.3) belong to  $\mathcal{C}_x^1(\mathbb{R}, \mathcal{H}_\zeta(\mathbb{C}^+))$  having continuous extensions to the real axis.*

**Proof.** Let introduce the dilated potential  $\mathcal{V}^h(x) = \mathcal{V}(hx + x_0)$ , defined with:  $x_0 = \frac{b+a}{2}$  and denote with  $\tilde{\chi}_\pm^h$  the Jost's solutions of the equation:  $(-\partial_x^2 + \mathcal{V}^h - \zeta^2)u = 0$ . The functions  $\chi_\pm^h$  and  $\tilde{\chi}_\pm^h$  are related by

$$\chi_\pm^h(x, \zeta, \mathcal{V}) = e^{\pm \frac{\zeta}{h} x_0} \tilde{\chi}_\pm^h\left(\frac{x - x_0}{h}, \zeta, \mathcal{V}\right). \quad (4.4)$$

When  $\mathcal{V}$  is defined according to (1.9), the corresponding dilated potential,  $\mathcal{V}^h$ , still satisfies a similar condition (with support over  $(-\frac{b-a}{2h}, \frac{b-a}{2h})$  w.r.t. to the new variable  $x' = \frac{x-x_0}{h}$ ). Thus, for fixed  $h > 0$ , the Proposition 2.1 applies to  $\tilde{\chi}_\pm^h$  which result to be  $\mathcal{C}_x^1(\mathbb{R}, \mathcal{H}_\zeta(\mathbb{C}^+))$  and has continuous extensions to the real axis. ■

It is worthwhile to notice that, in the attempt of extending our approach to this new setting, all the estimates involved in the proofs depend on  $h$  and exhibit exponential bounds w.r.t. to the small parameter. To fix this point, let  $h > 0$  and consider the functions  $\chi_\pm^h$ . The rescaled functions  $b_\pm^h(x, \zeta) = e^{\mp i \frac{\zeta}{h} x} \chi_\pm^h(x, \zeta)$  are defined through a Picard iteration procedure (see the e.g. in the eq. (2.38)). Taking into account the small- $h$  behaviour of the corresponding rescaled kernels, it results

$$\sup_{x \in \mathbb{R}, \zeta \in \mathbb{C}^+} |b_\pm(x, \zeta)| \leq e^{\frac{C_0}{h^2}}, \quad \sup_{x \in \mathbb{R}, \zeta \in \mathbb{C}^+} |b'_\pm(x, \zeta)| \leq e^{\frac{C_1}{h^2}}, \quad (4.5)$$

where the coefficients  $C_i$ ,  $i = 0, 1$ , possibly depend on the data  $a, b$ , and  $\|\mathcal{V}\|_{L^1(a,b)}$ . Then, a suitable rewriting of the Krein's formula (2.15) and of the results of the Lemmata 2.2 and 2.3 in the  $h$ -dependent case would allows to express the  $h$ -dependent waves operators as:  $\mathcal{W}_{\theta_1, \theta_2}^h = 1 + \mathcal{O}(h) + \mathcal{O}(h)$ , provided that

$$(\theta_1, \theta_2) \in \mathcal{B}_{\rho(h)}((0, 0)), \quad \text{with } \rho(h) = h e^{-\frac{\tilde{C}}{h^2}} \quad (4.6)$$

and  $\tilde{C} > 0$  large enough. Operators defined with the prescription (4.6) appear to be of small interest in the applications perspective. At this concern we recall that the adiabatic theorem obtained in [9] applies with:  $\theta_i = c_i h^{N_0}$ ,  $i = 1, 2$ , for some  $N_0 \in \mathbb{N}$  (see Theorem 7.1 in [9]). In this connection, it is important to relax the constraint expressed by (4.6) in order to obtain small- $h$  expansions of the waves operators, holding at least in a suitable subspace of  $L^2$ , when the parameters are assumed to be only polynomially smalls w.r.t.  $h$ .

#### 4.1 The Jost's solutions in the $h$ -dependent case.

Our purpose is to obtain small- $h$  estimates for the solutions of the equation

$$(-h^2 \partial_x^2 + \mathcal{V})u = k^2 u, \quad k \in \mathbb{R}, \quad (4.7)$$

under suitable restrictions on  $k^2$  and depending on the external conditions in  $\mathbb{R} \setminus (a, b)$ . We focus on the boundary value problem

$$\begin{cases} (-h^2 \partial_x^2 + \mathcal{V})u = k^2 u, & k \in \mathbb{R}, x \in (a, b), \\ (h \partial_x + i \gamma(k))u(a) = \ell_a, & (h \partial_x - i \gamma(k))u(b) = \ell_b. \end{cases} \quad (4.8)$$

where  $\gamma(\cdot)$  is real valued on  $\mathbb{R}$ , while  $\ell_a, \ell_b \in \mathbb{C}$ . In the following,  $\mathcal{V}$  is a potential barrier, defined according to the conditions (1.21), and  $c > 0$  is assumed to be such that the energy subset  $\Omega_c(\mathcal{V})$

$$\Omega_c(\mathcal{V}) = \{k^2 \in \mathbb{R}_+ \mid \inf \mathcal{V} - k^2 > c\}, \quad (4.9)$$

is not empty. The solutions of this problem are characterized as follows.

**Lemma 4.2** *Assume  $\mathcal{V}$  to fulfill the conditions (1.21),  $h > 0$  and  $k^2 \in \Omega_c(\mathcal{V})$  for a positive  $c$ . Then, for any  $(\ell_a, \ell_b) \in \mathbb{C}^2$ , the solution of (4.8) satisfies*

$$\sup_{x \in [a, b]} |u(x)| \leq C_{a,b,c} (|\ell_b| + |\ell_a|), \quad (4.10)$$

with  $C_{a,b,c}$  possibly dependent on the data.

**Proof.** Project the identity  $(-h^2 \partial_x^2 + \mathcal{V} - k^2)u = 0$  over  $u$ ; an integration by parts, taking into account the boundary conditions in (4.8), yields

$$\|hu'\|_{L^2(a,b)}^2 + \langle u, (\mathcal{V} - k^2)u \rangle_{L^2(a,b)} = h i \gamma(k) (|u(a)|^2 + |u(b)|^2) + h (\ell_a u^*(b) - \ell_b u^*(a)). \quad (4.11)$$

Since  $\gamma(\cdot)$  is real, the real part of the identity reads as

$$\|hu'\|_{L^2(a,b)}^2 + \langle u, (\mathcal{V} - k^2)u \rangle_{L^2(a,b)} = h \operatorname{Re} (\ell_a u^*(b) - \ell_b u^*(a)),$$

and, for  $k^2 \in \Omega_c(\mathcal{V})$ , we get

$$\|hu'\|_{L^2(a,b)}^2 + \|u\|_{L^2(a,b)}^2 \leq hC_c(|\ell_a| |u(b)| + |\ell_b| |u(a)|) .$$

Introducing the  $h$ -dependent Sobolev norm of  $u \in H^1(a,b)$

$$\|u\|_{H^{1,h}(a,b)}^2 = \|hu'\|_{L^2(a,b)}^2 + \|u\|_{L^2(a,b)}^2 , \quad (4.12)$$

the previous inequality rephrases as

$$\|u\|_{H^{1,h}(a,b)}^2 \leq hC_c(|\ell_a| |u(b)| + |\ell_b| |u(a)|) . \quad (4.13)$$

The Gagliardo-Nirenberg estimate  $\sup_{x \in [a,b]} |u(x)| \leq C_{b-a} \|u'\|_{L^2(a,b)}^{1/2} \|u\|_{L^2(a,b)}^{1/2}$  (e.g. in [21]) and the equivalence of  $\|u\|_{H^{1,h}(a,b)}$  with  $\|hu'\|_{L^2(a,b)} + \|u\|_{L^2(a,b)}$ , imply

$$\sup_{x \in [a,b]} |u(x)| \leq \frac{C_{b-a}}{2h^{1/2}} \|u\|_{H^{1,h}(a,b)} ; \quad (4.14)$$

using this inequality at the l.h.s. of (4.14), we get the result. ■

In what follows  $\psi_-^h(\cdot, k, \mathcal{V})$  denote the solutions of the equation (4.7) fulfilling the exterior conditions

$$\psi_-^h(x, k, \mathcal{V}) \Big|_{\substack{x < a \\ k > 0}} = e^{i\frac{k}{h}x} + R^h(k, \mathcal{V})e^{-i\frac{k}{h}x}, \quad \psi_-^h(x, k, \mathcal{V}) \Big|_{\substack{x > b \\ k > 0}} = T^h(k, \mathcal{V})e^{i\frac{k}{h}x}, \quad (4.15)$$

$$\psi_-^h(x, k, \mathcal{V}) \Big|_{\substack{x < a \\ k < 0}} = T^h(k, \mathcal{V})e^{i\frac{k}{h}x}, \quad \psi_-^h(x, k, \mathcal{V}) \Big|_{\substack{x > b \\ k < 0}} = e^{i\frac{k}{h}x} + R^h(k, \mathcal{V})e^{-i\frac{k}{h}x}, \quad (4.16)$$

while the functions  $G^{k,h}(\cdot, y, \mathcal{V})$  and  $H^{k,h}(\cdot, y, \mathcal{V})$  generalize, to the  $h$ -dependent case, the branch-cut limit of the Green's function and of its derivative (see the definition (2.31)). They are explicitly defined as solutions of the equation (4.7) in  $\mathbb{R} \setminus \{y\}$ , fulfilling the boundary conditions

$$G^{k,h}(y^+, y, \mathcal{V}) = G^{k,h}(y^-, y, \mathcal{V}), \quad h^2(\partial_1 G^{k,h}(y^+, y, \mathcal{V}) - \partial_1 G^{k,h}(y^-, y, \mathcal{V})) = -1, \quad (4.17)$$

$$h^2(H^{k,h}(y^+, y, \mathcal{V}) - H^{k,h}(y^-, y, \mathcal{V})) = 1, \quad \partial_1 H^{k,h}(y^+, y, \mathcal{V}) = \partial_1 H^{k,h}(y^-, y, \mathcal{V}). \quad (4.18)$$

**Proposition 4.3** *Let  $h, c > 0$ ,  $y \in \{a, b\}$  and  $j \in \{0, 1\}$ . Under the assumptions (1.21) and  $k^2 \in \Omega_c(\mathcal{V})$ , the relations*

$$\begin{aligned} \partial_1^j \chi_\pm^h(\cdot, k, \mathcal{V}) &= \mathcal{O}\left(\frac{1}{h^j}\right), & \partial_1^j \psi_-^h(\cdot, k, \mathcal{V}) &= \mathcal{O}\left(\frac{k}{h^j}\right), \\ G^{k,h}(\cdot, y, \mathcal{V}) &= \mathcal{O}\left(\frac{1}{h}\right), & H^{k,h}(\cdot, y, \mathcal{V}) &= \mathcal{O}\left(\frac{1}{h^2}\right), \end{aligned} \quad (4.19)$$

hold with the symbols  $\mathcal{O}(\cdot)$  referred to the metric space  $\mathbb{R} \times \{k \in \mathbb{R} \mid k^2 \in \Omega_c(\mathcal{V})\} \times \mathbb{R}_+$ .

**Proof.** To simplify the notations, in what follows the explicit dependence of  $\chi_\pm^h$ ,  $\psi_-^h$ ,  $G^{k,h}$  and  $H^{k,h}$  from the potential  $\mathcal{V}$  is omitted. We start considering the Jost's solutions  $\chi_\pm^h(\cdot, k)$ . Making use of the exterior conditions (4.3) and adapting the relations (2.22)-(2.23) to the  $h$ -dependent setting, it follows

$$\chi_+^h(\cdot, k) \Big|_{x < a} = \frac{h}{2ik} \left( (w_0^h(k))^* e^{-i\frac{k}{h}x} - w^h(k) e^{i\frac{k}{h}x} \right), \quad (4.20)$$

$$\chi_-^h(\cdot, k) \Big|_{x > b} = \frac{h}{2ik} \left( w_0^h(k) e^{i\frac{k}{h}x} - w^h(k) e^{-i\frac{k}{h}x} \right), \quad (4.21)$$

where  $w^h$  and  $w_0^h$  respectively denotes the Wronskians associated to the couples  $\{\chi_+^h(\cdot, k), \chi_-^h(\cdot, k)\}$  and  $\{\chi_+^h(\cdot, -k), \chi_-^h(\cdot, k)\}$ . As a consequence of (4.3), (4.20) and (4.21), the functions  $\chi_\pm^h(\cdot, k)$  are solutions of the problem (4.8) where  $\gamma(\cdot)$  is the identity, while:  $\ell_a = -hw^h(k)e^{i\frac{k}{h}a}$  and  $\ell_b = 0$  in the case of  $\chi_+^h(\cdot, k)$ , or  $\ell_a = 0$  and  $\ell_b = hw^h(k)e^{-i\frac{k}{h}b}$  in the case of  $\chi_-^h(\cdot, k)$ . Proceeding as in the proof of the Lemma 4.2, and taking into account the exterior conditions (4.3), we have

$$\|\chi_\pm^h(\cdot, k)\|_{H^{1,h}(a,b)}^2 \leq C_c h^2 |w^h(k)|. \quad (4.22)$$

According to the definition of  $w^h(k)$ , the equivalent representations hold

$$w^h(k) = e^{-i\frac{k}{h}a} \left( -i\frac{k}{h}\chi_+^h(a, k) - \partial_1\chi_+^h(a, k) \right), \quad (4.23)$$

$$w^h(k) = e^{i\frac{k}{h}b} \left( \partial_1\chi_-^h(b, k) - i\frac{k}{h}\chi_-^h(b, k) \right). \quad (4.24)$$

We use the  $h$ -dependent norms introduced in (4.12); from the relations (4.23)-(4.24) and the inequality (4.13), it follows

$$|w^h(k)| \leq \frac{|k|}{h} |\chi_+^h(a, k)| + |\partial_1\chi_+^h(a, k)| \lesssim \frac{|k|}{h^{3/2}} \|\chi_+^h(\cdot, k)\|_{H^{1,h}(a,b)} + \frac{1}{h^{1/2}} \|\partial_1\chi_+^h(\cdot, k)\|_{H^{1,h}(a,b)}, \quad (4.25)$$

and

$$|w^h(k)| \leq \frac{|k|}{h} |\chi_-^h(b, k)| + |\partial_1\chi_-^h(b, k)| \lesssim \frac{|k|}{h^{3/2}} \|\chi_-^h(\cdot, k)\|_{H^{1,h}(a,b)} + \frac{1}{h^{1/2}} \|\partial_1\chi_-^h(\cdot, k)\|_{H^{1,h}(a,b)}. \quad (4.26)$$

Exploiting the equivalence of  $\|u\|_{H^{1,h}(a,b)}$  with  $\|hu'\|_{L^2(a,b)} + \|u\|_{L^2(a,b)}$ , and using the identity:  $h\partial_1^2\chi_\pm^h = \frac{1}{h}(\mathcal{V} - k^2)\chi_\pm^h$ , we get

$$\|\partial_1\chi_\pm^h(\cdot, k)\|_{H^{1,h}(a,b)} \lesssim \|h\partial_1^2\chi_\pm^h(\cdot, k)\|_{L^2(a,b)} + \|\partial_1\chi_\pm^h(\cdot, k)\|_{L^2(a,b)} \lesssim \frac{1}{h} \|\chi_\pm^h(\cdot, k)\|_{L^2(a,b)} + \|\partial_1\chi_\pm^h(\cdot, k)\|_{L^2(a,b)},$$

which yields

$$\|\partial_1\chi_\pm^h(\cdot, k)\|_{H^{1,h}(a,b)} \lesssim 1/h \|\chi_\pm^h(\cdot, k)\|_{H^{1,h}(a,b)}; \quad (4.27)$$

replacing this inequality at the r.h.s. of (4.25) leads to

$$|w^h(k)| \lesssim \frac{1}{h^{3/2}} (1 + |k|) \|\chi_+^h(\cdot, k)\|_{H^{1,h}(a,b)}, \quad (4.28)$$

while, a similar estimate of  $|w^h(k)|$  in terms of the norm  $\|\chi_-^h(\cdot, k)\|_{H^{1,h}(a,b)}$ . Then, using (4.22) yields

$$\|\chi_\pm^h(\cdot, k)\|_{H^{1,h}(a,b)} \lesssim h^{1/2} (1 + |k|), \quad (4.29)$$

which entails

$$\sup_{x \in [a,b]} |\chi_\pm^h(x, k)| \lesssim (1 + |k|), \quad (4.30)$$

while, from (4.27), we get

$$\sup_{x \in [a,b]} |\partial_x \chi_\pm^h(x, k)| \lesssim \frac{1}{h} (1 + |k|). \quad (4.31)$$

In the exterior domain, the solutions are described by the relations (4.3) and (4.20)-(4.21). Then, to extend the above inequalities to the whole real axis, estimates for  $\chi_+^h$  and  $\chi_-^h$  in  $(-\infty, a)$  and  $(b, +\infty)$  respectively are needed. According to (4.28) and (4.29), it follows that:  $w^h(k) = \mathcal{O}(\frac{1}{h})$  uniformly w.r.t.  $k^2 \in \Omega_c(\mathcal{V})$ . Moreover, the representation

$$w_0^h(k) = \chi_+^h(a, -k) \partial_1\chi_-^h(a, k) - \partial_1\chi_+^h(a, -k) \chi_-^h(a, k), \quad (4.32)$$

and the previous estimates yield:  $w_0^h(k) = \mathcal{O}(\frac{1}{h})$ . As a by product of Proposition 4.1, for any  $x \in \mathbb{R}$ , the maps:  $k \rightarrow \chi_\pm^h(x, k)$ ,  $w^h(k)$  and  $w_0^h(k)$  are continuous including  $k = 0$ . This implies that the contributions at the r.h.s. of (4.20) and (4.21) behaves in  $k = 0$  as

$$w_0^h(k)e^{i\frac{k}{h}x} = a_0(h) + \mathcal{O}\left(\frac{k}{h}\right), \quad w^h(k)e^{-i\frac{k}{h}x} = a_0(h) + \mathcal{O}\left(\frac{k}{h}\right), \quad (4.33)$$

where  $a_0(h) = \mathcal{O}(\frac{1}{h}) \in \mathbb{R}$ . Therefore, the r.h.s. of (4.20) and (4.21) result uniformly bounded as  $h > 0$ ,  $k^2 \in \Omega_c(\mathcal{V})$  and  $x \in \mathbb{R}$ , while their derivatives w.r.t.  $x$  behaves as  $\mathcal{O}(\frac{k}{h})$ .

Next consider the functions  $\psi_-^h(\cdot, k)$ ,  $G^{k,h}(\cdot, y)$ ,  $H^{k,h}(\cdot, y)$ , with  $y \in \{a, b\}$ ; these express in terms of the Jost's solutions  $\chi_\pm^h$  according to

$$\psi_-^h(x, k) = \begin{cases} -\frac{2ik}{hw^h(k)}\chi_+^h(x, k), & \text{for } k \geq 0, \\ \frac{2ik}{hw^h(-k)}\chi_-^h(x, -k), & \text{for } k < 0. \end{cases}$$



and

$$G^{k,h}(\cdot, y) = \frac{1}{h^2 w^h(k)} \begin{cases} \chi_+^h(\cdot, k) \chi_-^h(\cdot, k), & x \geq y, \\ \chi_-^h(\cdot, k) \chi_+^h(\cdot, k), & x < y, \end{cases} \quad H^{k,h}(\cdot, y) = \frac{-1}{h^2 w^h(k)} \begin{cases} \chi_+^h(\cdot, k) \partial_1 \chi_-^h(\cdot, k), & x \geq y, \\ \chi_-^h(\cdot, k) \partial_1 \chi_+^h(\cdot, k), & x < y. \end{cases}$$

Using the relations

$$\chi_+^h(\cdot, k) = \frac{h}{2ik} \left( (w_0^h(k))^* \chi_-^h(\cdot, k) - w^h(k) \chi_-^h(\cdot, -k) \right), \quad (4.34)$$

$$\chi_-^h(\cdot, k) = \frac{h}{2ik} \left( w_0^h(k) \chi_+^h(\cdot, k) - w^h(k) \chi_+^h(\cdot, -k) \right), \quad (4.35)$$

the identity

$$|w^h(k)|^2 = \frac{k^2}{h^2} + |w_0^h(k)|^2 \quad (4.36)$$

follows for  $k \in \mathbb{R}$  and  $h > 0$ . This entails:  $|w^h(k)|^{-1} \leq \frac{h}{|k|}$ . Then, the terms  $\frac{\pm 2ik}{hw^h(\mp k)}$  are bounded uniformly w.r.t.  $h > 0$  and  $k \in \mathbb{R}$ , while  $\frac{1}{h^2 w^h(k)} = \mathcal{O}\left(\frac{1}{hk}\right)$ . Thus, the relations in (4.19) concerned with  $\psi_-^h(\cdot, k)$ ,  $G^{k,h}(\cdot, y)$  and  $H^{k,h}(\cdot, y)$  are straightforward consequences of the characterization of  $\chi_\pm^h$ , obtained above. ■

Next we focus on the case where the potential, depending on  $h$ , is formed by the superpositions of a potential barrier,  $\mathcal{V}$ , plus a selfadjoint term,  $\mathcal{W}^h$ , supported inside  $(a, b)$  with support of size  $h$ . Namely, it is assumed that  $\mathcal{V}^h = \mathcal{V} + \mathcal{W}^h$ , with  $\mathcal{V}$  a potential barrier fulfilling (1.21), while  $\mathcal{W}^h$  is defined according to (1.19)-(1.20). The functions  $\chi_\pm^h(x, k, \mathcal{V}^h)$  are defined by the equations (4.2) and the exterior conditions (4.3). In particular, (4.2) rephrases in terms of the integral equation

$$\chi_\pm^h(x, k, \mathcal{V}^h) = \chi_\pm^h(x, k, \mathcal{V}) - \frac{1}{h^2} \int_x^{x_\pm} \mathcal{K}^h(t, x, k, \mathcal{V}) \mathcal{W}^h(t) \chi_\pm^h(t, k, \mathcal{V}^h) dt, \quad (4.37)$$

where the boundary points are fixed by:  $x_+ = b$  and  $x_- = a$ , while  $x < x_+$  or  $x > x_-$  depending on the case which is being considered. The kernel

$$\mathcal{K}^h(t, x, k, \mathcal{V}) = \frac{\chi_+^h(t, k, \mathcal{V}) \chi_-^h(x, k, \mathcal{V}) - \chi_-^h(t, k, \mathcal{V}) \chi_+^h(x, k, \mathcal{V})}{w^h(k, \mathcal{V})}, \quad (4.38)$$

is defined with  $w^h(k, \mathcal{V})$  denoting the Jost's function associated with the couple  $\{\chi_+^h(\cdot, k, \mathcal{V}), \chi_-^h(\cdot, k, \mathcal{V})\}$ ; namely we have

$$w^h(k, \mathcal{V}) = w(\chi_+^h(\cdot, k, \mathcal{V}), \chi_-^h(\cdot, k, \mathcal{V})). \quad (4.39)$$

This framework allows to discuss the properties of  $\chi_\pm^h(x, k, \mathcal{V}^h)$  by making use of the results obtained in the case when  $\mathcal{W}^h = 0$ .

**Proposition 4.4** *Let  $h_0, c > 0$ ,  $h \in (0, h_0)$  and  $\mathcal{V}^h = \mathcal{V} + \mathcal{W}^h$  defined according to the conditions (1.19)-(1.20) and (1.21)-(1.22). Assuming that  $k^2 \in \Omega_c(\mathcal{V})$ , the relations*

$$\chi_\pm^h(x, k, \mathcal{V}^h) = \mathcal{O}(1), \quad \partial_x \chi_\pm^h(x, k, \mathcal{V}^h) = \mathcal{O}\left(\frac{1}{h}\right), \quad (w^h(k, \mathcal{V}^h))^{-1} = \mathcal{O}(h), \quad (4.40)$$

hold with  $\mathcal{O}(\cdot)$  referred to the metric space  $\mathbb{R} \times \{k \in \mathbb{R} \mid k^2 \in \Omega_c(\mathcal{V})\} \times (0, h_0)$ .

**Proof.** In the  $h$ -dependent case, the functions  $G^{k,h}$ ,  $H^{k,h}$  and  $\psi_-^h$  are represented in terms of the Jost's solutions  $\chi_\pm^h$  as

$$G^{k,h}(\cdot, y, \mathcal{V}) = \frac{1}{h^2 w^h(k, \mathcal{V})} \begin{cases} \chi_+^h(\cdot, k, \mathcal{V}) \chi_-^h(\cdot, k, \mathcal{V}), & x \geq y, \\ \chi_-^h(\cdot, k, \mathcal{V}) \chi_+^h(\cdot, k, \mathcal{V}), & x < y, \end{cases} \quad (4.41)$$

$$-H^{k,h}(\cdot, y, \mathcal{V}) = \frac{1}{h^2 w^h(k, \mathcal{V})} \begin{cases} \chi_+^h(\cdot, k, \mathcal{V}) \partial_1 \chi_-^h(\cdot, k, \mathcal{V}), & x \geq y, \\ \chi_-^h(\cdot, k, \mathcal{V}) \partial_1 \chi_+^h(\cdot, k, \mathcal{V}), & x < y, \end{cases} \quad (4.42)$$

and

$$\psi_-^h(\cdot, k, \mathcal{V}) = \begin{cases} -\frac{2ik}{hw^h(k, \mathcal{V})} \chi_+^h(\cdot, k, \mathcal{V}), & \text{for } k \geq 0, \\ \frac{2ik}{hw^h(-k, \mathcal{V})} \chi_-^h(\cdot, -k, \mathcal{V}), & \text{for } k < 0. \end{cases} \quad (4.43)$$

Next, assume the potential  $\mathcal{V}$  fulfilling the conditions (1.21) and, for a fixed  $x$ , consider the maps  $\mathcal{K}^h(\cdot, x, k, \mathcal{V})$ . We have

$$\begin{aligned} 1_{\{t \geq x\}}(t) \mathcal{K}^h(t, x, k, \mathcal{V}) &= \begin{cases} h^2 G^{k,h}(t, x, \mathcal{V}) + \frac{h}{2ik} \psi_-^h(x, k, \mathcal{V}) \chi_-^h(t, k, \mathcal{V}), & \text{for } k \geq 0, \\ h^2 G^{k,h}(t, x, \mathcal{V}) + \frac{h}{2ik} (\psi_-^h(t, k, \mathcal{V}))^* \chi_+^h(x, k, \mathcal{V}), & \text{for } k < 0, \end{cases} \\ 1_{\{t < x\}}(t) \mathcal{K}^h(t, x, k, \mathcal{V}) &= \begin{cases} -\frac{h}{2ik} \psi_-^h(t, k, \mathcal{V}) \chi_-^h(x, k, \mathcal{V}) - h^2 G^{k,h}(t, x, \mathcal{V}), & \text{for } k \geq 0, \\ -\frac{h}{2ik} (\psi_-^h(x, k, \mathcal{V}))^* \chi_+^h(t, k, \mathcal{V}) - h^2 G^{k,h}(t, x, \mathcal{V}), & \text{for } k < 0, \end{cases} \end{aligned}$$

where the relations:  $(\chi_{\pm}^h(\cdot, k, \mathcal{V}))^* = \chi_{\pm}^h(\cdot, -k, \mathcal{V})$  are taken into account. Exploiting the above identities, it follows

$$1_{\{t \geq x\}}(t) \partial_x \mathcal{K}^h(t, x, k, \mathcal{V}) = \begin{cases} -h^2 H^{k,h}(t, x, \mathcal{V}) + \frac{h}{2ik} \partial_x \psi_-^h(x, k, \mathcal{V}) \chi_-^h(t, k, \mathcal{V}), & \text{for } k \geq 0, \\ -h^2 H^{k,h}(t, x, \mathcal{V}) + \frac{h}{2ik} (\psi_-^h(t, k, \mathcal{V}))^* \partial_x \chi_+^h(x, k, \mathcal{V}), & \text{for } k < 0, \end{cases}$$

and

$$1_{\{t < x\}}(t) \mathcal{K}^h(t, x, k, \mathcal{V}) = \begin{cases} -\frac{h}{2ik} \psi_-^h(t, k, \mathcal{V}) \partial_x \chi_-^h(x, k, \mathcal{V}) + h^2 H^{k,h}(t, x, \mathcal{V}), & \text{for } k \geq 0, \\ -\frac{h}{2ik} (\partial_x \psi_-^h(x, k, \mathcal{V}))^* \chi_+^h(t, k, \mathcal{V}) + h^2 H^{k,h}(t, x, \mathcal{V}), & \text{for } k < 0. \end{cases}$$

In all cases, from the result of the Proposition 4.3, we get

$$\partial_x^j \mathcal{K}^h(t, x, k, \mathcal{V}) = \mathcal{O}(h^{1-j}), \quad j = 0, 1, \quad (4.44)$$

uniformly w.r.t.  $(t, x) \in \mathbb{R}^2$  and  $k : k^2 \in \Omega_c(\mathcal{V})$ .

Next, consider the function  $\chi_+^h(\cdot, k, \mathcal{V}^h)$ ; For  $x < b$ ,  $\chi_+^h(\cdot, k, \mathcal{V}^h)$  solves the equation (4.37) with  $x_+ = b$ . We look for the solution of this problem in the form of a Picard's series:  $\chi_+^h(\cdot, k, \mathcal{V}^h) = \sum_{n=0}^{+\infty} \chi_{+,n}^h(\cdot, k, \mathcal{V}^h)$

$$\chi_{+,0}^h(\cdot, k, \mathcal{V}^h) = \chi_+^h(\cdot, k, \mathcal{V}), \quad \chi_{+,n}^h(x, k, \mathcal{V}^h) = -\frac{1}{h^2} \int_x^b \mathcal{K}^h(t, x, k, \mathcal{V}) \mathcal{W}^h(t) \chi_{+,n-1}^h(t, k, \mathcal{V}^h) dt. \quad (4.45)$$

Due to our assumptions and to results of the Propositions 4.1 and 4.3, the first term of this expansion is continuous in  $x$  and  $k$ , and bounded according to the relations (4.19); in what follows we set

$$M(c, \mathcal{V}) = \sup_{x \in \mathbb{R}, k^2 \in \Omega_c(\mathcal{V}), h \in (0, h_0)} |\chi_{+,0}^h(\cdot, k, \mathcal{V}^h)|. \quad (4.46)$$

The second contribution is given by

$$\chi_{+,1}^h(x, k, \mathcal{V}^h) = -\frac{1}{h^2} \int_x^b \mathcal{K}^h(t, x, k, \mathcal{V}) \mathcal{W}^h(t) \chi_{+,0}^h(t, k, \mathcal{V}^h) dt; \quad (4.47)$$

due to the regularity of the kernel  $\mathcal{K}^h$  and  $\chi_{+,0}^h$ , this is a continuous function w.r.t.  $x$  and  $k$ , while from the relation (4.44), it follows

$$|\chi_{+,1}^h(x, k, \mathcal{V}^h)| \leq M(c, \mathcal{V}) F(x, c, \mathcal{V}), \quad (4.48)$$

where

$$F(x, h, c, \mathcal{V}) = \frac{C(c, \mathcal{V})}{h} \int_x^b |\mathcal{W}^h(t)| dt, \quad C(c, \mathcal{V}) = \frac{1}{h} \sup_{\substack{t, x \in \mathbb{R} \\ k^2 \in \Omega_c(\mathcal{V})}} |\mathcal{K}^h(t, x, k, \mathcal{V})|. \quad (4.49)$$

Next, assume  $\chi_{+,n-1}^h(x, k, \mathcal{V}^h)$  to be continuous w.r.t.  $x$  and  $k$  fulfilling the inequality

$$|\chi_{+,n-1}^h(x, k, \mathcal{V}^h)| \leq M(c, \mathcal{V}) \frac{F^{n-1}(x, h, c, \mathcal{V})}{(n-1)!}, \quad (4.50)$$

and consider the term  $\chi_{+,n}^h(x, k, \mathcal{V}^h)$ ; due to the equation (4.45), this is still continuous w.r.t.  $x$  and  $k$  allowing the estimate

$$\begin{aligned} |\chi_{+,n}^h(x, k, \mathcal{V}^h)| &\leq \frac{1}{h^2} \int_x^b |\mathcal{K}^h(t, x, k, \mathcal{V}) \mathcal{W}^h(t) \chi_{+,n-1}^h(t, k, \mathcal{V}^h)| dt \\ &\leq \frac{M(c, \mathcal{V}) C(c, \mathcal{V})}{h} \int_x^b |\mathcal{W}^h(t)| \frac{F^{n-1}(t, h, c, \mathcal{V})}{(n-1)!} dt = -\frac{M(c, \mathcal{V})}{n!} \int_x^b \partial_t F^n(t, h, c, \mathcal{V}) dt \\ &= \frac{M(c, \mathcal{V})}{n!} F^n(x, h, c, \mathcal{V}) \end{aligned}$$

Then, an induction argument shows that the Picard's series uniformly converges to  $\chi_+^h$  and

$$\sup_{x < b, k^2 \in \Omega_c} |\chi_+^h(x, k, \mathcal{V}^h)| \leq M(c, \mathcal{V}) e^{\|F(\cdot, h, c, \mathcal{V})\|_{L^\infty(a, b)}}. \quad (4.51)$$

Since

$$\|F(\cdot, h, c, \mathcal{V})\|_{L^\infty(a, b)} \lesssim \frac{1}{h} \|\mathcal{W}^h\|_{L^1(a, b)} \lesssim 1, \quad (4.52)$$

it follows

$$\sup_{x < b, k^2 \in \Omega_c} |\chi_+^h(x, k, \mathcal{V}^h)| \lesssim 1 \Rightarrow 1_{\{x < b\}}(x) 1_{\Omega_c}(k^2) \chi_+^h(x, k, \mathcal{V}^h) = \mathcal{O}(1). \quad (4.53)$$

The first relation in (4.40) is a consequence of (4.53) and of the exterior condition  $\chi_+^h(\cdot, k, \mathcal{V})|_{x > b} = e^{i\frac{k}{h}x}$ .

Next, consider  $\partial_1 \chi_+^h$ ; for  $x \geq b$  it is explicitly defined by  $i\frac{k}{h}e^{i\frac{k}{h}x}$ , while, for  $x < b$ , it fulfills the equation

$$\partial_x \chi_+^h(x, k, \mathcal{V}^h) = \partial_x \chi_+^h(x, k, \mathcal{V}) - \frac{1}{h^2} \int_x^b \partial_x \mathcal{K}^h(t, x, k, \mathcal{V}) \mathcal{W}^h(t) \chi_+^h(t, k, \mathcal{V}^h) dt. \quad (4.54)$$

Being  $\chi_+^h$  and  $\partial_x \mathcal{K}^h$  uniformly bounded for  $k^2 \in \Omega_c$ , the integral part at the r.h.s. of (4.54) results dominated  $\mathcal{O}(1/h)$  and, according to the result of Proposition 4.3, the same holds for the function  $\partial_x \chi_+^h(x, k, \mathcal{V})$ . It follows

$$1_{\{x < b\}}(x) 1_{\Omega_c}(k^2) \partial_x \chi_+^h(x, k, \mathcal{V}^h) = \mathcal{O}\left(\frac{1}{h}\right). \quad (4.55)$$

This relations and the explicit form  $\partial_x \chi_+^h = i\frac{k}{h}e^{i\frac{k}{h}x}$ , holding for  $x > b$ , yields the second relation in (4.40).

The proof, in the case of the function  $\chi_-^h$ , is obtained following the same line.

The above result, and a straightforward adaptation of the inequality (4.25) to the  $h$ -dependent case, yield, for  $k^2 \in \Omega_c(\mathcal{V})$ , the relation

$$|k| \leq h w^h(k, \mathcal{V}^h) \lesssim 1. \quad (4.56)$$

being  $w^h(k, \mathcal{V}^h)$  the Jost's function associated to the couple  $\chi_\pm^h(x, k, \mathcal{V}^h)$ . Proceeding as in the Lemma 2.2, we write  $w^h(k, \mathcal{V}^h)$  as

$$w^h(k, \mathcal{V}^h) = -\partial_1 b_+^h(a, k) - 2i\frac{k}{h} b_+^h(a, k), \quad (4.57)$$

where  $b_+^h(x, k) = e^{-i\frac{k}{h}x} \chi_+^h(x, k, \mathcal{V}^h)$  identifies with the Picard series:  $b_+^h = \sum_{n=0}^{+\infty} b_{+,n}^h$ , whose terms are defined, for  $x < b$ , by:  $b_{+,0}^h(x, \zeta) = 1$  and

$$b_{+,n}^h(x, \zeta) = \frac{1}{h} \int_x^b \frac{e^{2i\frac{k}{h}(t-x)} - 1}{2ik} \mathcal{V}^h(t) b_{+,n-1}^h(t, k) dt, \quad n \in \mathbb{N}^*, \quad x < b. \quad (4.58)$$

In the limit  $k \rightarrow 0$ , we get

$$b_{+,n}^h(x, 0) = \frac{1}{h^2} \int_x^b (t-x) \mathcal{V}^h(t) b_{+,n-1}^h(t, 0) dt, \quad n \in \mathbb{N}^*, \quad x < b. \quad (4.59)$$

Using the assumption (1.22), and  $b_{+,0}^h = 1$ , an induction argument leads to:  $b_{+,n}^h(x, 0) \geq 0$  and  $b_+^h(x, 0) \geq 1$ . The function  $\partial_1 b_+^h(x, 0)$  is defined by

$$\partial_1 b_+^h(x, 0) = -\frac{1}{h^2} \int_x^b \mathcal{V}^h(t) b_+^h(t, 0) dt, \quad x < b. \quad (4.60)$$

Since  $b_+^h > 0$  and  $\mathcal{V}^h > 0$ , at least in a subset of  $(a, b)$ , it follows:  $\partial_1 b_+^h(a, 0) < 0$ . Then, computing  $w^h(0, \mathcal{V}^h)$  through the relation (4.57) gives:  $w^h(0, \mathcal{V}^h) = \partial_1 b_+^h(a, k) \neq 0$ . According to the previous results, the continuous map  $k \rightarrow hw^h(k, \mathcal{V}^h)$  has no zeroes and a positive  $C$  exists s.t.

$$hw^h(k, \mathcal{V}^h) > C.$$

This leads us to the last relation in (4.40). ■

## 4.2 Generalized eigenfunctions expansions.

In what follows, we use the maps  $\Gamma_{i=1,2}$ , the matrices  $A_{\theta_1, \theta_2}$ ,  $B_{\theta_1, \theta_2}$  and the functions  $G^{k,h}$ ,  $H^{k,h}$  (referred to the notation introduced in the first part of this Section; see eqs. (2.10), (2.11)-(2.12)) and (4.41)-(4.42)). For  $\mathcal{U} \in L^\infty(\mathbb{R}, \mathbb{R})$ , with  $\text{supp } \mathcal{U} = (a, b)$ , let consider the operator  $Q^h(\mathcal{U})$

$$\begin{cases} \mathcal{D}(Q^h(\mathcal{U})) = H^2(\mathbb{R} \setminus \{a, b\}), \\ (Q^h(\mathcal{U})u)(x) = -h^2 u''(x) + \mathcal{U}(x)u(x), \quad \text{for } x \in \mathbb{R} \setminus \{a, b\}, \end{cases} \quad (4.61)$$

and the map:  $\Gamma_1^h = h^2 \Gamma_1$ . According to the definition given in the Section 2, the set  $\{\mathbb{C}^4, \Gamma_1^h, \Gamma_2\}$  forms a boundary triple for  $Q^h(\mathcal{U})$ ; using this framework, the operator  $Q_{\theta_1, \theta_2}^h(\mathcal{U})$ , introduced in (1.18), identifies with the restriction

$$Q_{\theta_1, \theta_2}^h(\mathcal{U}) : \begin{cases} \mathcal{D}(Q_{\theta_1, \theta_2}^h(\mathcal{U})) = \left\{ u \in D(Q^h(\mathcal{U})) \mid A_{\theta_1, \theta_2}^h \Gamma_1^h u = B_{\theta_1, \theta_2} \Gamma_2 u \right\}, \\ Q_{\theta_1, \theta_2}^h(\mathcal{U})u = Q^h(\mathcal{U})u. \end{cases} \quad (4.62)$$

parametrized by the  $\mathbb{C}^{4,4}$ -block-diagonal matrices  $A_{\theta_1, \theta_2}^h = 1/h^2 A_{\theta_1, \theta_2}$  and  $B_{\theta_1, \theta_2}$ . Let  $\psi_-^h(\cdot, k, \theta_1, \theta_2, \mathcal{U})$  denotes the generalized eigenfunctions of the operator  $Q_{\theta_1, \theta_2}^h(\mathcal{U})$ , describing an incoming wave function of momentum  $k$ ; this is a solutions of the boundary value problem

$$\begin{cases} (-h^2 \partial_x^2 + \mathcal{U})u = k^2 u, & x \in \mathbb{R} \setminus \{a, b\}, \quad k \in \mathbb{R}, \\ A_{\theta_1, \theta_2}^h \Gamma_1^h u = B_{\theta_1, \theta_2} \Gamma_2 u, \end{cases}$$

fulfilling exterior conditions of the type (4.15)-(4.16). The result of the Proposition 2.5 can be formally rephrased within this  $h$ -dependent setting by expressing the difference:  $\psi_-^h(\cdot, k, \theta_1, \theta_2, \mathcal{U}) - \psi_-^h(\cdot, k, \mathcal{U})$  in terms of the functions  $g(e_i, k, \mathcal{U})$

$$\{g^h(e_i, k, \mathcal{U})\}_{i=1}^4 = \{G^{k,h}(\cdot, b, \mathcal{U}), -H^{k,h}(\cdot, b, \mathcal{U}), G^{k,h}(\cdot, a, \mathcal{U}), -G^{k,h}(\cdot, a, \mathcal{U})\}, \quad (4.63)$$

according to

$$\begin{aligned} & \psi_-^h(\cdot, k, \theta_1, \theta_2, \mathcal{U}) \\ &= \begin{cases} \psi_-^h(\cdot, k, \mathcal{U}) - \sum_{i,j=1}^4 \left[ (\mathcal{M}^h(k, \theta_1, \theta_2, \mathcal{U}))^{-1} B_{\theta_1, \theta_2} \right]_{ij} [\Gamma_2 \psi_-^h(\cdot, k, \mathcal{U})]_j g^h(e_i, k, \mathcal{U}), & \text{for } k \geq 0, \\ \psi_-^h(\cdot, k, \mathcal{U}) - \sum_{i,j=1}^4 \left[ (\mathcal{M}^h(-k, \theta_1, \theta_2, \mathcal{U}))^{-1} B_{\theta_1, \theta_2} \right]_{ij} [\Gamma_2 \psi_-^h(\cdot, k, \mathcal{U})]_j g^h(e_i, -k, \mathcal{U}), & \text{for } k < 0, \end{cases} \end{aligned} \quad (4.64)$$

This relation makes sense as far as  $k$  is not a singular point of the matrix  $\mathcal{M}^h(k, \theta_1, \theta_2)$ , which is defined by

$$\mathcal{M}^h(k, \theta_1, \theta_2, \mathcal{U}) = (B_{\theta_1, \theta_2} q^h(k, \mathcal{U}) - A_{\theta_1, \theta_2}^h), \quad (4.65)$$

with

$$q^h(k, \mathcal{U}) = \begin{pmatrix} G^{k,h}(b, b, \mathcal{U}) & \frac{1}{2} - H^{k,h}(b^+, b, \mathcal{U}) & G^{k,h}(b, a, \mathcal{U}) & -H^{k,h}(b, a, \mathcal{U}) \\ H^{k,h}(b^+, b, \mathcal{U}) - \frac{1}{2} & -\partial_1 H^{k,h}(b, b, \mathcal{U}) & H^{k,h}(a, b, \mathcal{U}) & -\partial_1 H^{k,h}(b, a, \mathcal{U}) \\ G^{k,h}(a, b, \mathcal{U}) & -H^{k,h}(a, b, \mathcal{U}) & G^{k,h}(a, a, \mathcal{U}) & -\left(\frac{1}{2} + H^{k,h}(a^-, a, \mathcal{U})\right) \\ H^{k,h}(b, a, \mathcal{U}) & -\partial_1 H^{k,h}(a, b, \mathcal{U}) & H^{k,h}(a^-, a, \mathcal{U}) + \frac{1}{2} & -\partial_1 H^{k,h}(a, a, \mathcal{U}) \end{pmatrix}. \quad (4.66)$$

(we refer for this point to the definitions (2.51) and (4.41)-(4.42)).

Next, we consider the case where the potential  $\mathcal{U}$  coincides with:  $\mathcal{V}^h = \mathcal{V} + \mathcal{W}^h$  defined according to the conditions (1.21), (1.19)-(1.20). Assume that  $\Omega_c(\mathcal{V}) \neq \emptyset$  for  $c > 0$ ; when  $k^2 \in \Omega_c(\mathcal{V})$ , the result of Proposition 4.4 applies to the Jost's solutions  $\chi_+^h(\cdot, k, \mathcal{V}^h)$ . Then, from the representations (4.41)-(4.42), follows

$$G^{k,h}(x, y, \mathcal{V}^h) = \mathcal{O}\left(\frac{1}{h}\right), \quad H^{k,h}(x, y, \mathcal{V}^h) = \mathcal{O}\left(\frac{1}{h^2}\right), \quad \partial_1 H^\zeta(x, y) = \mathcal{O}\left(\frac{1}{h^3}\right). \quad (4.67)$$

Taking into account (4.67), the matrix  $\mathcal{M}^h(k, \theta_1, \theta_2, \mathcal{V}^h)$  explicitly writes as

$$\begin{aligned} 1_{\Omega_c(\mathcal{V})}(k^2) \mathcal{M}^h(k, \theta_1, \theta_2, \mathcal{V}^h) = & -\frac{1}{h^2} \begin{pmatrix} \alpha(\theta_2) & & & \\ & \alpha(\theta_1) & & \\ & & \alpha(-\theta_2) & \\ & & & \alpha(-\theta_1) \end{pmatrix} \\ & + \begin{pmatrix} \beta(\theta_2) \mathcal{O}(1 + \frac{1}{h^2}) & \beta(\theta_2) \mathcal{O}(\frac{1}{h^3}) & \beta(\theta_2) \mathcal{O}(\frac{1}{h^2}) & \beta(\theta_2) \mathcal{O}(\frac{1}{h^3}) \\ \beta(\theta_1) \mathcal{O}(\frac{1}{h}) & \beta(\theta_1) \mathcal{O}(1 + \frac{1}{h^2}) & \beta(\theta_1) \mathcal{O}(\frac{1}{h}) & \beta(\theta_1) \mathcal{O}(\frac{1}{h^2}) \\ \beta(-\theta_2) \mathcal{O}(\frac{1}{h^2}) & \beta(-\theta_2) \mathcal{O}(\frac{1}{h^3}) & \beta(-\theta_2) \mathcal{O}(1 + \frac{1}{h^2}) & \beta(-\theta_2) \mathcal{O}(\frac{1}{h^3}) \\ \beta(-\theta_1) \mathcal{O}(\frac{1}{h}) & \beta(-\theta_1) \mathcal{O}(\frac{1}{h^2}) & \beta(-\theta_1) \mathcal{O}(\frac{1}{h}) & \beta(-\theta_1) \mathcal{O}(1 + \frac{1}{h^2}) \end{pmatrix} \end{aligned} \quad (4.68)$$

where the holomorphic functions  $\alpha(\cdot)$  and  $\beta(\cdot)$  are defined in (2.53). Then a direct computation gives

$$\det 1_{\Omega_c(\mathcal{V})}(k^2) \mathcal{M}^h(k, \theta_1, \theta_2, \mathcal{V}^h) = \frac{1}{h^8} \det(A_{\theta_1, \theta_2}) + \mathcal{O}\left(\frac{\theta_1}{h^8}\right) + \mathcal{O}\left(\frac{\theta_2}{h^8}\right),$$

$$\det(-A_{\theta_1, \theta_2}) = 4 \left(1 + \cosh \frac{\theta_1}{2}\right) \left(1 + \cosh \frac{\theta_2}{2}\right).$$

and the condition  $(\theta_1, \theta_2) \in \mathcal{B}_\delta((0, 0))$ , with  $\delta$  suitably small (independently from  $h$ !), entails

$$\det 1_{\Omega_c(\mathcal{V})}(k^2) \mathcal{M}^h(k, \theta_1, \theta_2, \mathcal{V}^h) \gtrsim 1/h^8. \quad (4.69)$$

As a consequence,  $\mathcal{M}^h$  is invertible whenever  $(\theta_1, \theta_2)$  is close to the origin and  $k^2 \in \Omega_c(\mathcal{V})$ . In these conditions, the inverse matrix writes as

$$\begin{aligned} 1_{\Omega_c(\mathcal{V})}(k^2) (\mathcal{M}^h(k, \theta_1, \theta_2, \mathcal{V}^h))^{-1} \\ = \frac{1}{\det 1_{\Omega_c(\mathcal{V})}(k^2) \mathcal{M}^h(k, \theta_1, \theta_2, \mathcal{V}^h)} \left[ \frac{1}{h^8} \det(A_{\theta_1, \theta_2}) \text{diag}\{\lambda_i^h\} + M^h(k, \theta_1, \theta_2, \mathcal{V}^h) \right], \end{aligned} \quad (4.70)$$

where the main term in (2.66),  $\text{diag}(\lambda_i)$ , is the  $\mathbb{C}^{4,4}$  diagonal matrix defined by the coefficients

$$\{\lambda_i\}_{i=1}^4 = \left\{ \frac{-h^2}{\alpha(\theta_2)}, \frac{-h^2}{\alpha(\theta_1)}, \frac{-h^2}{\alpha(-\theta_2)}, \frac{-h^2}{\alpha(-\theta_1)} \right\}, \quad (4.71)$$

while the remainder is

$$M^h(k, \theta_1, \theta_2, \mathcal{V}^h) = \frac{1}{h^8} \begin{pmatrix} \mathcal{O}(\theta_1) + \mathcal{O}(\theta_2) & \mathcal{O}(\theta_2) & \mathcal{O}(\theta_2) & \mathcal{O}(\theta_2) \\ \mathcal{O}(\theta_1) & \mathcal{O}(\theta_1) + \mathcal{O}(\theta_2) & \mathcal{O}(\theta_1) & \mathcal{O}(\theta_1) \\ \mathcal{O}(\theta_2) & \mathcal{O}(\theta_2) & \mathcal{O}(\theta_1) + \mathcal{O}(\theta_2) & \mathcal{O}(\theta_2) \\ \mathcal{O}(\theta_1) & \mathcal{O}(\theta_1) & \mathcal{O}(\theta_1) & \mathcal{O}(\theta_1) + \mathcal{O}(\theta_2) \end{pmatrix}, \quad (4.72)$$

From (4.69), it follows that the matrix coefficients in (4.70) are uniformly bounded w.r.t.  $k^2 \in \Omega_c(\mathcal{V})$ ,  $(\theta_1, \theta_2) \in \mathcal{B}_\delta((0, 0))$  and  $h \in (0, h_0)$  for suitable (and possibly small)  $c$ ,  $\delta$  and  $h_0$ . Exploiting this result, the relations (4.64) rephrase as follows

$$\begin{aligned} 1_{\Omega_c(\mathcal{V})}(k^2) [\psi_-^h(\cdot, k, \theta_1, \theta_2, \mathcal{V}^h) - \psi_-^h(\cdot, k, \mathcal{V}^h)] \\ = 1_{\Omega_c(\mathcal{V})}(k^2) [\mathcal{O}(\theta_2) G^{\sigma k, h}(\cdot, b, \mathcal{V}^h) + \mathcal{O}(\theta_1) H^{\sigma k, h}(\cdot, b, \mathcal{V}^h) + \mathcal{O}(\theta_2) G^{\sigma k, h}(\cdot, a, \mathcal{V}^h) + \mathcal{O}(\theta_1) H^{\sigma k, h}(\cdot, a, \mathcal{V}^h)] \end{aligned} \quad (4.73)$$

with:  $\sigma = \frac{k}{|k|}$ . It is worthwhile to notice that, being the matrix coefficients in  $\mathcal{M}^h(k, \theta_1, \theta_2, \mathcal{V}^h)$  and  $B_{\theta_1, \theta_2}$  holomorphic w.r.t. the parameters  $(\theta_1, \theta_2)$ , under the above conditions the same holds for the coefficients of  $(\mathcal{M}^h(k, \theta_1, \theta_2, \mathcal{U}))^{-1} B_{\theta_1, \theta_2}$ . Then, the symbols  $\mathcal{O}(\cdot)$  in (4.73), dependent from the variables  $\{k, \theta_1, \theta_2, h\}$  and defined in the sense of the metric space  $\mathbb{R} \times \mathcal{B}_\delta((0, 0)) \times (0, h_0)$ , denote holomorphic functions of  $\theta_1$  and  $\theta_2$ .

### 4.3 Proof of the Theorem 1.3.

Proceeding as in Section 3, we introduce the generalized Fourier transform associated to  $Q_{0,0}^h(\mathcal{V}^h)$

$$(\mathcal{F}_{\mathcal{V}^h}^h \varphi)(k) = \int_{\mathbb{R}} \frac{dx}{(2\pi h)^{1/2}} (\psi_-^h(x, k, \mathcal{V}^h))^* \varphi(x), \quad \varphi \in L^2(\mathbb{R}). \quad (4.74)$$

According to our assumptions on the potential, this is an unitary map with range  $L^2(\mathbb{R})$  and  $(\mathcal{F}_{\mathcal{V}^h}^h)^{-1}$  acts as

$$(\mathcal{F}_{\mathcal{V}^h}^h)^{-1} f(x) = \int \frac{dk}{(2\pi h)^{1/2}} \psi_-^h(x, k, \mathcal{V}^h) f(k) \quad (4.75)$$

for all  $f \in L^2(\mathbb{R})$ . Let consider the operator  $\mathcal{W}_{\theta_1, \theta_2, c}^h$  defined by the integral kernel

$$\mathcal{W}_{\theta_1, \theta_2, c}^h(x, y) = \int_{\mathbb{R}} \frac{dk}{2\pi h} 1_{\Omega_c(\mathcal{V})}(k^2) \psi_-^h(x, k, \theta_1, \theta_2, \mathcal{V}^h) (\psi_-^h(y, k, \mathcal{V}^h))^* . \quad (4.76)$$

Assume the parameters  $\theta_1, \theta_2$  to be close enough to the origin, so that expansion (4.73) holds; the action of  $\mathcal{W}_{\theta_1, \theta_2, c}^h$  over  $\varphi \in L^2(\mathbb{R})$ , writes as

$$\begin{aligned} \mathcal{W}_{\theta_1, \theta_2, c}^h \varphi &= \int_{\mathbb{R}} \frac{dk}{(2\pi h)^{1/2}} 1_{\Omega_c(\mathcal{V})}(k^2) \psi_-^h(x, k, \mathcal{V}^h) (\mathcal{F}_{\mathcal{V}^h}^h \varphi)(k) \\ &+ \int_{\mathbb{R}} \frac{dk}{(2\pi h)^{1/2}} 1_{\Omega_c(\mathcal{V})}(k^2) [\mathcal{O}(\theta_2) G^{\sigma k, h}(\cdot, b, \mathcal{V}^h) + \mathcal{O}(\theta_2) G^{\sigma k, h}(\cdot, a, \mathcal{V}^h)] (\mathcal{F}_{\mathcal{V}^h}^h \varphi)(k) \\ &+ \int_{\mathbb{R}} \frac{dk}{(2\pi h)^{1/2}} 1_{\Omega_c(\mathcal{V})}(k^2) [\mathcal{O}(\theta_1) H^{\sigma k, h}(\cdot, b, \mathcal{V}^h) + \mathcal{O}(\theta_1) H^{\sigma k, h}(\cdot, a, \mathcal{V}^h)] (\mathcal{F}_{\mathcal{V}^h}^h \varphi)(k). \end{aligned} \quad (4.77)$$

with  $\sigma k = |k|$ . The first contribution to this expansion identifies with the the spectral projector over the energy subspace  $\Omega_c(\mathcal{V})$ . Making use of the notation

$$\Pi_{\Omega_c(\mathcal{V})} \varphi = \int_{\mathbb{R}} \frac{dk}{(2\pi h)^{1/2}} 1_{\Omega_c(\mathcal{V})}(k^2) \psi_-^h(x, k, \mathcal{V}^h) (\mathcal{F}_{\mathcal{V}^h}^h \varphi)(k), \quad (4.78)$$

we get

$$(\mathcal{W}_{\theta_1, \theta_2, c}^h - \Pi_{\Omega_c(\mathcal{V})}) \varphi = \sum_{\alpha=a, b} [\phi_{\alpha}^h + \psi_{\alpha}^h], \quad (4.79)$$

where

$$\phi_{\alpha}^h(x) = \int_{\mathbb{R}} \frac{dk}{(2\pi h)^{1/2}} \mathcal{O}(\theta_2) 1_{\Omega_c(\mathcal{V})}(k^2) G^{|k|, h}(x, \alpha, \mathcal{V}^h) (\mathcal{F}_{\mathcal{V}^h}^h \varphi)(k), \quad \alpha \in \{a, b\}, \quad (4.80)$$

$$\psi_{\alpha}^h(x) = \int_{\mathbb{R}} \frac{dk}{(2\pi h)^{1/2}} \mathcal{O}(\theta_1) 1_{\Omega_c(\mathcal{V})}(k^2) H^{|k|, h}(x, \alpha, \mathcal{V}^h) (\mathcal{F}_{\mathcal{V}^h}^h \varphi)(k), \quad \alpha \in \{a, b\}. \quad (4.81)$$

Starting from (4.79)-(4.81), the proof follows the same line as in the Proposition 3.1. In particular, making use of the representation of  $G^{k, h}$  and  $H^{k, h}$  given in (4.41)-(4.42), the sum at the r.h.s. of (4.79) writes as

$$\sum_{\alpha=a, b} [\phi_{\alpha}^h(x) + \psi_{\alpha}^h(x)] = \quad (4.82)$$

$$\begin{aligned} &1_{\{x \leq a\}}(x) (\mathcal{F}_0^h)^{-1} (1_{k < 0}(k) 1_{\Omega_c(\mathcal{V})}(k^2) \mu_1^h(k) \mathcal{F}_{\mathcal{V}^h}^h \varphi(k) + \mathcal{P}(1_{k > 0}(k) 1_{\Omega_c(\mathcal{V})}(k^2) \mu_1^h(k) \mathcal{F}_{\mathcal{V}^h}^h \varphi(k)))(x) \\ &+ 1_{\{x \geq b\}}(x) (\mathcal{F}_0^h)^{-1} (\mathcal{P}(1_{k < 0}(k) 1_{\Omega_c(\mathcal{V})}(k^2) \mu_2^h(k) \mathcal{F}_{\mathcal{V}^h}^h \varphi(k) + 1_{k > 0}(k) 1_{\Omega_c(\mathcal{V})}(k^2) \mu_2^h(k) \mathcal{F}_{\mathcal{V}^h}^h \varphi(k)))(x) \\ &+ 1_{(a, b)}(x) (\mathcal{F}_{\mathcal{V}^h}^h)^{-1} (1_{k < 0}(k) 1_{\Omega_c(\mathcal{V})}(k^2) \mu_3^h(k) \mathcal{F}_{\mathcal{V}^h}^h \varphi(k) - \mathcal{P}(1_{k > 0}(k) 1_{\Omega_c(\mathcal{V})}(k^2) \mu_4^h(k) \mathcal{F}_{\mathcal{V}^h}^h \varphi(k)))(x), \end{aligned}$$

where the auxiliary functions  $\mu_{i=1,2,3,4}^h$  allow the representation:  $1_{\Omega_c(\mathcal{V})}(k^2) \mu_{i=1,2,3,4}^h(k) = \mathcal{O}(\frac{\theta_1}{h^2}) + \mathcal{O}(\frac{\theta_2}{h^2})$ . Since  $(\mathcal{F}_{\mathcal{V}^h}^h)^{\pm 1}$  are unitary maps, it follows

$$\left\| \sum_{\alpha=a, b} [\phi_{\alpha}^h(x) + \psi_{\alpha}^h(x)] \right\|_{L^2(\mathbb{R})} \lesssim \left( \frac{|\theta_1|}{h^2} + \frac{|\theta_2|}{h^2} \right) \|\varphi\|_{L^2(\mathbb{R})}, \quad (4.83)$$

which entails

$$\|\mathcal{W}_{\theta_1, \theta_2, c}^h - \Pi_{\Omega_c(\mathcal{V})}\|_{\mathcal{L}(L^2(\mathbb{R}), L^2(\mathbb{R}))} = \mathcal{O}\left(\frac{\theta_1}{h^2}\right) + \mathcal{O}\left(\frac{\theta_2}{h^2}\right). \quad (4.84)$$

Denote with  $\mathcal{H}_{\Omega_c(\mathcal{V})}$  the projection space  $\mathcal{H}_{\Omega_c(\mathcal{V})} = \Pi_{\Omega_c(\mathcal{V})}L^2(\mathbb{R})$ ; the map  $\mathcal{W}_{\theta_1, \theta_2, c}^h$  acts on  $\mathcal{H}_{\Omega_c(\mathcal{V})}$  as

$$\mathcal{W}_{\theta_1, \theta_2, c}^h \psi = \psi + \mathcal{O}\left(\frac{\theta_1}{h^2}\right) + \mathcal{O}\left(\frac{\theta_2}{h^2}\right), \quad \psi \in \mathcal{H}_{\Omega_c(\mathcal{V})}. \quad (4.85)$$

According to this relation, the restriction  $\mathcal{W}_{\theta_1, \theta_2, c}^h|_{\mathcal{H}_{\Omega_c(\mathcal{V})}}$  is an invertible map from the range  $R\left(\mathcal{W}_{\theta_1, \theta_2, c}^h|_{\mathcal{H}_{\Omega_c(\mathcal{V})}}\right)$  in  $\mathcal{H}_{\Omega_c(\mathcal{V})}$ . We consider the action of the dynamical system generated by  $-iQ_{\theta_1, \theta_2}^h(\mathcal{V}^h)$  over the space  $R\left(\mathcal{W}_{\theta_1, \theta_2, c}^h|_{\mathcal{H}_{\Omega_c(\mathcal{V})}}\right)$ . Let  $\psi \in \mathcal{H}_{\Omega_c(\mathcal{V})}$  be the unique solution of the equation

$$\mathcal{W}_{\theta_1, \theta_2, c}^h|_{\mathcal{H}_{\Omega_c(\mathcal{V})}} \psi = u, \quad u \in R\left(\mathcal{W}_{\theta_1, \theta_2, c}^h|_{\mathcal{H}_{\Omega_c(\mathcal{V})}}\right). \quad (4.86)$$

Due to the intertwining property

$$Q_{\theta_1, \theta_2}^h(\mathcal{V}^h)\mathcal{W}_{\theta_1, \theta_2, c}^h = \mathcal{W}_{\theta_1, \theta_2, c}^h Q_{0,0}^h(\mathcal{V}^h), \quad (4.87)$$

we have

$$\begin{aligned} e^{-itQ_{\theta_1, \theta_2}^h(\mathcal{V}^h)}u &= \mathcal{W}_{\theta_1, \theta_2, c}^h e^{-itQ_{0,0}^h(\mathcal{V}^h)}\left(\mathcal{W}_{\theta_1, \theta_2, c}^h|_{\mathcal{H}_{\Omega_c(\mathcal{V})}}\right)^{-1}u \\ &= \mathcal{W}_{\theta_1, \theta_2, c}^h e^{-itQ_{0,0}^h(\mathcal{V}^h)}\psi. \end{aligned} \quad (4.88)$$

Since  $e^{-itQ_{0,0}^h(\mathcal{V}^h)}$  is strongly continuous in time and  $\mathcal{W}_{\theta_1, \theta_2, c}^h$  forms an analytic family of type A w.r.t.  $(\theta_1, \theta_2)$ , the same hold for  $e^{-itQ_{\theta_1, \theta_2}^h(\mathcal{V}^h)}\Pi_{\Omega_c(\mathcal{V})}$ . Moreover, using the relations:  $\psi = \Pi_{\Omega_c(\mathcal{V})}\psi$  and  $\Pi_{\Omega_c(\mathcal{V})}e^{-itQ_{0,0}^h(\mathcal{V}^h)}\Pi_{\Omega_c(\mathcal{V})} = e^{-itQ_{0,0}^h(\mathcal{V}^h)}\Pi_{\Omega_c(\mathcal{V})}$  and the operator expansion induced by (4.84), it follows

$$e^{-itQ_{\theta_1, \theta_2}^h(\mathcal{V}^h)}\mathcal{W}_{\theta_1, \theta_2, c}^h\Pi_{\Omega_c(\mathcal{V})}\psi = \Pi_{\Omega_c(\mathcal{V})}e^{-itQ_{0,0}^h(\mathcal{V}^h)}\Pi_{\Omega_c(\mathcal{V})}\psi + \left(\mathcal{O}\left(\frac{\theta_1}{h^2}\right) + \mathcal{O}\left(\frac{\theta_2}{h^2}\right)\right)\psi, \quad (4.89)$$

which implies

$$\left(e^{-itQ_{\theta_1, \theta_2}^h(\mathcal{V}^h)}\mathcal{W}_{\theta_1, \theta_2, c}^h - e^{-itQ_{0,0}^h(\mathcal{V}^h)}\right)\Pi_{\Omega_c(\mathcal{V})} = \mathcal{O}\left(\frac{\theta_1}{h^2}\right) + \mathcal{O}\left(\frac{\theta_2}{h^2}\right). \quad (4.90)$$

Here the symbols  $\mathcal{O}(\theta_i/h^2)$  denote bounded operators on  $L^2$  whose norm is dominated by  $|\theta_i|/h^2$ . Since we are interested in the action of the propagator  $e^{-itQ_{\theta_1, \theta_2}^h(\mathcal{V}^h)}$  over the spectral subspace  $\mathcal{H}_{\Omega_c(\mathcal{V})}$ , let consider in detail the first contribution at the r.h.s. of (4.90). From the expansion (4.79), (4.80)-(4.81) it follows

$$e^{-itQ_{\theta_1, \theta_2}^h(\mathcal{V}^h)}\left(\mathcal{W}_{\theta_1, \theta_2, c}^h - \Pi_{\Omega_c(\mathcal{V})}\right)\varphi = e^{-itQ_{\theta_1, \theta_2}^h(\mathcal{V}^h)}\sum_{\alpha=a,b}[\phi_\alpha^h + \psi_\alpha^h], \quad (4.91)$$

with  $\varphi \in L^2$ . Using (4.90) and  $\mathcal{W}_{\theta_1, \theta_2, c}^h = \mathcal{W}_{\theta_1, \theta_2, c}^h\Pi_{\Omega_c(\mathcal{V})}$ , we get

$$\left(e^{-itQ_{0,0}^h(\mathcal{V}^h)} - e^{-itQ_{\theta_1, \theta_2}^h(\mathcal{V}^h)}\right)\Pi_{\Omega_c(\mathcal{V})}\varphi = e^{-itQ_{\theta_1, \theta_2}^h(\mathcal{V}^h)}\sum_{\alpha=a,b}[\phi_\alpha^h + \psi_\alpha^h] + \left(\mathcal{O}\left(\frac{\theta_1}{h^2}\right) + \mathcal{O}\left(\frac{\theta_2}{h^2}\right)\right)\varphi. \quad (4.92)$$

Since  $Q_{\theta_1, \theta_2}^h(\mathcal{V}^h)$  is a restriction of  $Q^h(\mathcal{V}^h)$  and  $G^{k,h}$ ,  $H^{k,h}$  represent the limit of the defect functions in the kernel  $\ker(Q^h(\mathcal{V}^h) - z)$  as  $z$  tends to the branch cut, the operator  $e^{-itQ_{\theta_1, \theta_2}^h(\mathcal{V}^h)}$  formally acts on  $G^{k,h}$ ,  $H^{k,h}$  as

$$e^{-itQ_{\theta_1, \theta_2}^h(\mathcal{V}^h)}\Psi = e^{-itQ^h(\mathcal{V}^h)}\Psi = e^{-itk^2}\Psi, \quad \Psi = G^{k,h}, H^{k,h}.$$

Then, taking into account the explicit form of  $\phi_\alpha^h$ ,  $\psi_\alpha^h$ , we get

$$e^{-itQ_{\theta_1, \theta_2}^h(\mathcal{V}^h)}\phi_\alpha^h(x) = \int_{\mathbb{R}} \frac{dk}{(2\pi h)^{1/2}} \mathcal{O}(\theta_2) 1_{\Omega_c(\mathcal{V})}(k^2) e^{-itk^2} G^{|k|,h}(x, \alpha, \mathcal{V}^h) (\mathcal{F}_{\mathcal{V}^h}^h \varphi)(k), \quad \alpha \in \{a, b\}, \quad (4.93)$$

$$e^{-itQ_{\theta_1, \theta_2}^h(\mathcal{V}^h)}\psi_\alpha^h(x) = \int_{\mathbb{R}} \frac{dk}{(2\pi h)^{1/2}} \mathcal{O}(\theta_1) 1_{\Omega_c(\mathcal{V})}(k^2) e^{-itk^2} H^{|k|,h}(x, \alpha, \mathcal{V}^h) (\mathcal{F}_{\mathcal{V}^h}^h \varphi)(k), \quad \alpha \in \{a, b\}. \quad (4.94)$$

Proceeding as in the case  $\phi_\alpha^h, \psi_\alpha^h$ , similar expansions to the ones given in (4.82) is obtained for the r.h.s. of (4.93)-(4.94). It results

$$\left\| e^{-itQ_{\theta_1, \theta_2}^h(\mathcal{V}^h)} \sum_{\alpha=a, b} [\phi_\alpha^h(x) + \psi_\alpha^h(x)] \right\|_{L^2(\mathbb{R})} \lesssim \left( \frac{|\theta_1|}{h^2} + \frac{|\theta_2|}{h^2} \right) \|\varphi\|_{L^2(\mathbb{R})} \quad (4.95)$$

Therefore, (4.92) rephrases as

$$\left( e^{-itQ_{0,0}^h(\mathcal{V}^h)} - e^{-itQ_{\theta_1, \theta_2}^h(\mathcal{V}^h)} \right) \Pi_{\Omega_c(\mathcal{V})} = \mathcal{O}\left(\frac{\theta_1}{h^2}\right) + \mathcal{O}\left(\frac{\theta_2}{h^2}\right). \quad (4.96)$$

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## References

- [1] J. Aguilar, J.M. Combes. A class of analytic perturbations for one-body Schrödinger Hamiltonians. *Comm. Math. Phys.*, **22**, 269–279, 1971.
- [2] E. Balslev, J.M. Combes. Spectral properties of many-body Schrödinger operators with dilatation-analytic interactions. *Comm. Math. Phys.*, **22**, 280–294, 1971.
- [3] J. Behrndt, M.M. Malamud, H. Neidhardt. Scattering matrices and Weyl functions. *Proc. Lond. Math. Soc.* **97**, no.3, 568–598, 2008.
- [4] V. Bonnaillie-Noël, F. Nier, Y. Patel. Far from equilibrium steady states of 1D-Schrödinger-Poisson systems with quantum wells I. *Ann. I.H.P. An. Non Linéaire*, **25**, 937-968, 2008.
- [5] V. Bonnaillie-Noël, F. Nier, Y. Patel. Far from equilibrium steady states of 1D-Schrödinger-Poisson systems with quantum wells II. *J. Math. Soc. of Japan.*, **61**, 65-106, 2009.
- [6] V. Bonnaillie-Noël, F. Nier, Y. Patel. Computing the steady states for an asymptotic model of quantum transport in resonant heterostructures. *Journal of Computational Physics*, **219**(2), 644-670, 2006.
- [7] M. Brown, M. Marletta, S. Naboko, I. Wood. Boundary triplets and M-functions for non-selfadjoint operators, with applications to elliptic PDEs and block operator matrices. *J. Lond. Math. Soc.* (2) **77** no. 3, 700–718, 2008.
- [8] V.A. Derkach, M. M. Malamud. Generalized resolvents and the boundary value problems for Hermitian operators with gaps, *J. Funct. Anal.* 95 (1991), 1-95.
- [9] A. Faraĵ, A. Mantile, F. Nier. Adiabatic evolution of 1D shape resonances: an artificial interface conditions approach. *M3AS*, **21** no. 3, 541-618, 2011.
- [10] A. Faraĵ, A. Mantile, F. Nier. An explicit model for the adiabatic evolution of quantum observables driven by 1D shape resonances. *J. Phys. A: Math. Theor.* **43** (2010).
- [11] B. Helffer. *Semiclassical analysis for the Schrödinger operator and applications*, volume 1336 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1988.
- [12] B. Helffer, J. Sjöstrand. *Résonances en limite semi-classique*. *Mém. Soc. Mat. France (N.S.)*, number 24-25, 1986.
- [13] G. Jona-Lasionio, C. Presilla, J. Sjöstrand. On the Schrödinger equation with concentrated non linearities. *Ann. Physics*, **240**(1), 1-21, 1995.
- [14] T. Kato. Wave Operators and Similarity for Some Non-Selfadjoint Operators. *Math. Annalen*, 162, 258-279, 1966.
- [15] D. Krejčířík, P. Siegl, J. Železný. On the similarity of Sturm-Liouville operators with non-Hermitian boundary conditions to self-adjoint and normal operators. *To appear on Complex Anal. Oper. Theory*. Preprint: arXiv:1108.4946v1.



- [16] V.E. Lyantze, O.G. Storozh. *Methods of the theory of unbounded operators*. Naukova Dumka, Kiev, 1983.
- [17] M.M. Malamud, V.I. Mogilevskii. On extensions of dual pairs of operators. *Dopov. Nats. Akad. Nauk Ukr. Mat. Prirodozn. Tekh. Nauki*, no. 1, 30–37, 1997.
- [18] M.M. Malamud, V.I. Mogilevskii. On Weyl functions and Q-functions of dual pairs of linear relations. *Dopov. Nats. Akad. Nauk Ukr. Mat. Prirodozn. Tekh. Nauki*, no. 4, 32–37, 1999.
- [19] M.M. Malamud, V.I. Mogilevskii. Kreĭn type formula for canonical resolvents of dual pairs of linear relations. *Methods Funct. Anal. Topology*, **8**, no. 4, 72–100, 2002.
- [20] G. Nenciu, Linear adiabatic theory. Exponential estimates. *Comm. Math. Phys.*, **152**(3); 479–496, 1993.
- [21] L. Nirenberg, On elliptic partial differential equations, *Ann. Scuola Norm. Sup. Pisa*, **13**, 116–162, 1959.
- [22] K. Pankrashkin, Resolvents of self-adjoint extensions with mixed boundary conditions. *Rep. Math. Phys.*, **58**(2), 207–221, 2006.
- [23] C. Presilla, J. Sjöstrand. Transport properties in resonant tunneling heterostructures. *J. Math. Phys.*, **37**(10), 4816–4844, 1996.
- [24] C. Presilla, J. Sjöstrand. Nonlinear resonant tunneling in systems coupled to quantum reservoirs. *Phys. Rev. B: Condensed matter*, **55** no15, 9310–9313, 1997.
- [25] M. Reed, B. Simon. *Methods of modern mathematical physics vol. IV: Analysis of Operators*. Academic Press, New York, 1978.
- [26] V.A. Ryzhov. On a singular and an absolutely continuous subspace of a nonselfadjoint operator whose characteristic function has boundary values on the real axis. *Funct. Anal. Appl.* **32**, no. 3, 208–212, 1998.
- [27] V.A. Ryzhov. Equipped absolutely continuous subspaces and stationary construction of the wave operators in the non-self-adjoint scattering theory. *J. Math. Sci.*, **85**, no. 2, 1997.
- [28] V.A. Ryzhov, Functional model of a class of non-selfadjoint extensions of symmetric operators. *Oper. Theory Adv. Appl.*, vol. 174, 117–158, 2007.
- [29] D.R. Yafaev. *Mathematical Scattering Theory: Analytic theory*. Mathematical Surveys and Monographs vol. 158, American Mathematical Society, Providence, 2010.
- [30] D.R. Yafaev. *Mathematical Scattering Theory: General Theory*. American Mathematical Society, Providence, 1992.